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Meteorology. — *"An instrument to be used by the pilot of an aeroplane to measure the vertical velocity of the machine"*. By Dr. C. SCHOUTE, F. A. VAN HEYST, and N. E. GROENEVELD MEYER. (Communicated by Dr. VAN DER STOK).

(Communicated in the meeting of June 29, 1918).

In many cases it is important that the pilot of an aeroplane should get direct information about the velocity of climbing or falling of his machine, and in the greater part of these cases it is of a higher value that he should know the vertical speed with respect to the medium, the air, than in relation to the earth. The latter especially when he wishes to know the velocity of ascent or descent to be able to judge the effect of his altitude-steerage, as well in relation to the safety of flight as where either an extreme quickness or an extreme slowness of the climbing or falling is required.

As a rule the vertical speed is determined by measuring the quickness with which the atmospheric pressure varies. Therefore barometric readings can be executed after certain intervals. In that case however we do not get instantaneous values. The quickness of the changes in the pressure of the air can also be measured by means of the so called variometer, consisting of a vessel that communicates with the outer air by a narrow opening or a capillary tube and is provided with another tube in which a liquid-column can move to and fro when notwithstanding the "leakage" an under-pressure or an over-pressure is generated in the vessel. All these determinations based on pressure-changes give the vertical speed in relation to the earth and not to the medium, and moreover the results are not independent of the height at which they are obtained, so that generally a correction, and often a considerable correction, has to be applied.

This complication does not arise when the determination is made by vertical cup-anemometers. The idea suggests itself, to get indications of the vertical speed electrically by means of a dynamo-armature mounted on the axis of a vertical anemometer. The more so because indications at a distance are wanted, as the anemometer requires to be mounted in undisturbed air, which as a rule is not to be expected immediately in front of the pilot-seat. In such measurings however, the result would be spoiled by the friction, which is unavoidable in the anemometer, and which cannot be slight in this

case, because of the strong lateral pressure on the axis during the rapid motion. The rapport between the vertical speed V_v, and the number of revolutions N can be expressed by the relation

$$V_v = A + B\,N$$

in which A and B are constant. In the case V_v is smaller than A the anemometer stands still. Further it would already be a difficulty to obtain a permanent vertical position of the axis.

In the instrument the description of which follows and which we have called *scansimeter* the foresaid disadvantages of the vertical anemometer have been avoided without the advantages having been abandoned. Therefore electric currents are measured which are generated in a dynamo armature, mounted on an axis which has the direction of the line of displacement of the machine instead of being kept vertical. This directing as well as the rotation of the axis are obtained by an airscrew fixed on the hind part of the axis, which is made to rotate by reaction during the rapid motion through the air, and thus gives the correct position to the well-poised axis. Therefore it was practically sufficient to fix the axis pivoting around a line perpendicular to the body of the machine, which line must be horizontal when the aeroplane lies flat in a lateral sense. This simplification was applied for the advantage in the construction only. In what follows this line is always dealt with as a horizontal one, which does not introduce an appreciable error.

The total speed of the machine V_t can, in perfect anology to what has been said about the vertical velocity be represented by

$$V_t = A + BN$$

In this case the influence of the constant of friction A is much smaller because V_t is always considerable, so that when the axis is made to run lightly between ballbearings the number of revolutions N is approximately proportional to V_t.

In order to generate a current which offers a measure, not of the total speed with respect to the air, but of the vertical component of that quantity, $V_v = V_t \sin \alpha$ (fig. 1), the magnetic field in which the dynamo-coil rotates is dressed in a special way, viz. in such a way that the direction of the field, which we suppose to be homogeneous for the sake of simplicity, is always kept horizontal and besides that parallel to the vertical plane through the axis of rotation.

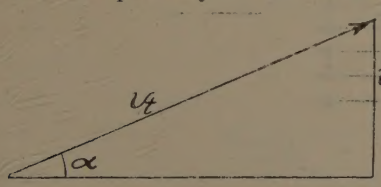

Fig 1

In order to realise this in the first model slits have been made in the permanent magnet, which was to procure the magnetic force. This magnet was suspended between horizontal pivots, in such a way that the lines of force were always kept horizontal. By inserting the axis of rotation through the slits, this axis can be made to be parallel to the lines of force when the direction of the flight is horizontal, while when the machine climbs or falls the angle between the lines of force and the axis corresponds to α in fig. 1.

The later models had two magnets instead of one, placed parallel to each other on both sides of the axis. In both cases the horizontal axis of suspension of the magnet was made to coincide with the line around which the armature axis pivoted.

If the axis AA (fig. 2) be parallel to the lines of force, no difference of potential will arise between the ends of the rotating

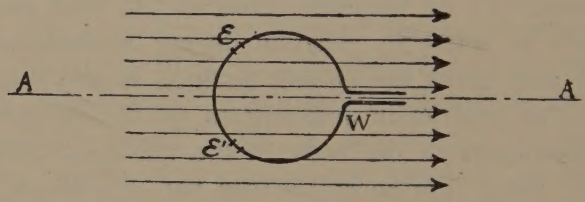

Fig 2

winding W. The electromotive force in an element ε of the coil is exactly counteracted by an equivalent E. M. F. in the opposite sense, generated in the element ε' lying in the position of a reflected image with respect to the axis.

If the axis AA (fig. 3) makes an angle α with the lines of force, an electromotive force will be generated in W proportional not only

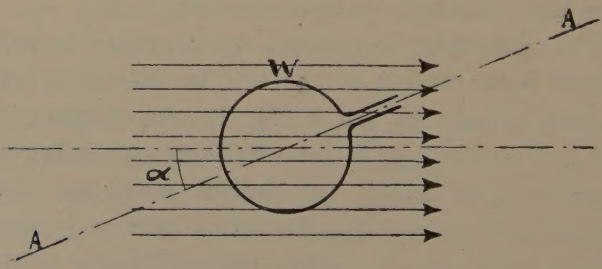

Fig 3

to the speed of rotation and the strength of the field, but caeteris paribus also to $\sin \alpha$. The magnetic field can be considered as being

the effect of a cooperation of two fields, one in the direction of the axis of rotation and another parallel to the vertical plane through the axis and perpendicular to the latter, which two fields bear a proportion as for the strength as $\cos \alpha : \sin \alpha$. Of those fields only the latter can generate a current in W.

When the changes in the electromotive force in W in consequence of changes in α shall be proportional to $\sin \alpha$, the condition must be made, that the magnetic field be unvariable when the relative position of axis and lines of force changes. Therefore 1 the material for the axis of rotation must be a non-magnetic one, and 2 if the homogeneity is not to be relied upon, and if a weak iron core shall be used, the form of the core must be globular.

By combining the use of a dynamo axis driven by reaction by an air screw with the *sinusdynamo* described here in principle, a system is obtained by which currents are generated, proportional as well to the total speed V_t of fig. 1, as to the sinus of the angle α under which the machine climbs or falls, (fig. 3) so that the strength of those currents offers a direct measure of the vertical velocity with respect to the medium, which measure is independent of the height of observation and of all quantities connected with the machine, so that no correction whatever has to be applied.

In fig. 4. the mechanism of such a sinusdynamo is illustrated schematically. The armature has been executed in the most simple way: the coil W has been imbedded in two parallel circular grooves on both sides of the axis aa. In the magnet M the slits G allow

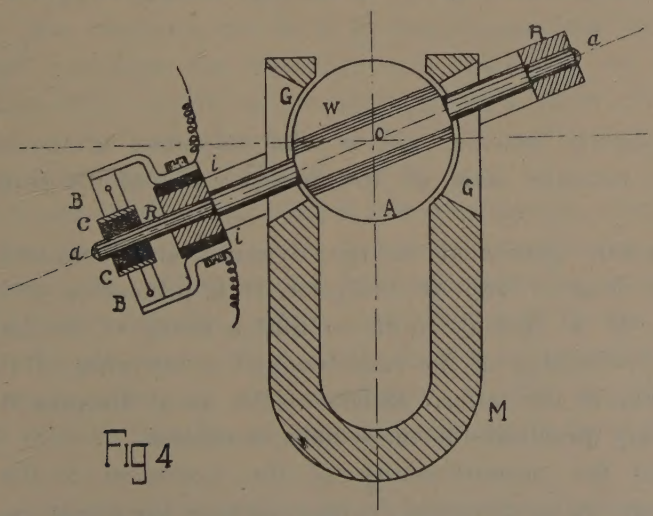

Fig 4

the axis to take different positions with respect to the magnet. The axis pivots around O, the point that at the same time determines

the horizontal axis on which the magnet is suspended. As by continuous currents only the direction of the motion is given, so that it is possible to discriminate between rising and falling, a two parted collector C has been made use of, with insulated (i) brushes B, fixed on the frame R.

Fig. 5 represents the scheme of the scansimeter.

Besides the parts noted in fig. 4 by analogous symbols this figure shows the screw S, the ball-bearings K and the circular cover Omh.

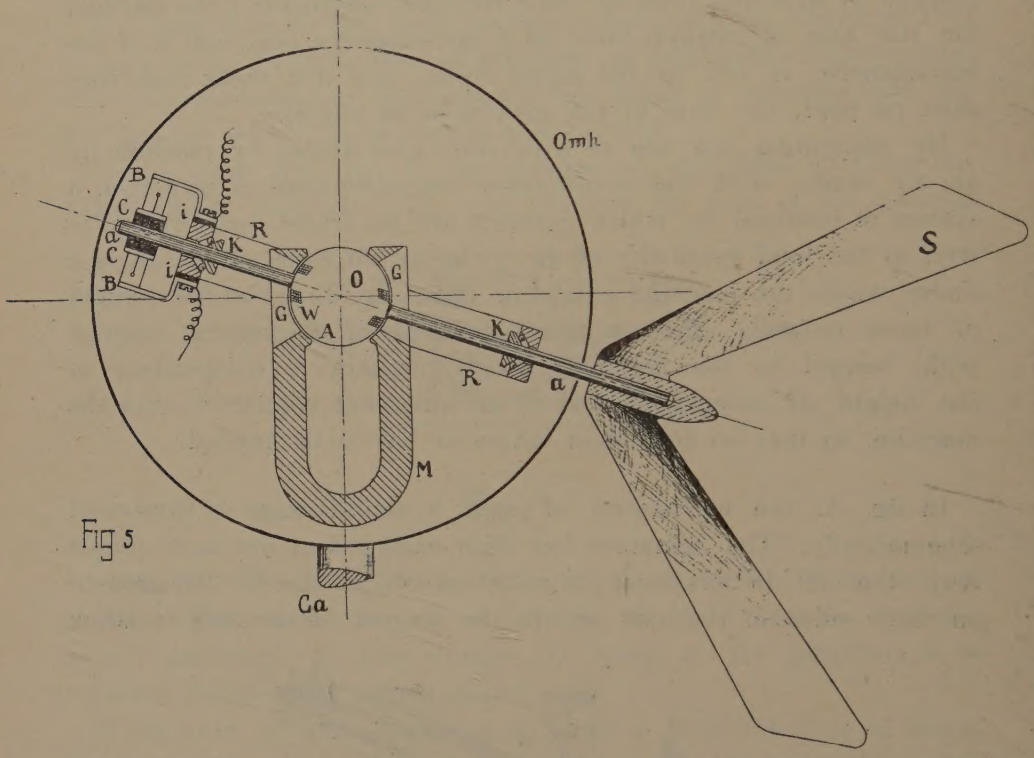

Fig 5

sideways slightly convex. Ga is what is shown of the support in which the common axis of the magnet and of the frame R are adjusted.

The currents generated in the dynamo are measured with an instrument, fixed in front of the pilot. It is preferable to choose an instrument of a special form, so that a rising of the hand corresponds to a climbing of the machine and a lowering of the former to a descent of the latter. Meters of the usual Desprez d'Arsonval type are very practical for these determinations.

To avoid the proportionality of the excursion to the speed of rotation from being disturbed by reaction from the armature, it is advisable to keep the current as weak as possible, consequently to give the measurings the character more of determinations of potential

differences than of the strength of currents, and therefore, to take a small resistance for the dynamo-coil in comparison to that of the current meter. On the other hand it is desirable to take a resistance for the dynamo coil of at least some times ten Ohm, in order to exclude a too great influence of transit-resistances.

The following can be recorded about the preparatory experiments carried out with this instrument.

One of the specimens of the scansimeter showed the following unexpected effect, which the others showed less or did not show at all. After the axis of rotation had been carefully placed horizontally in the laboratory, and the position of equilibrium of the magnet had been regulated in such a way that a rotation of the axis did not generate any current, a current could be produced arbitrarily in both directions by changes in the position of the brushes only. This phenomenon, rather startling at first sight, can be explained as follows [1]).

Evidently the magnetic field cannot be considered as resulting from two fields, one after the axis and another perpendicular to the axis, but here a third component has to be accepted, which we can assume to be horizontal and perpendicular to the axis. If the brushes are fixed in such a position, that the component in the vertical plane comes to ful development, i. e. that the commutating of the current takes place at the moments the difference of potential is zero (fig. 6), the effect of the horizontal component vertical to the axis is likewise zero, because for this component the commutation comes at the moments the E. M. F. has its maximal value (fig. 7). When the windings are distorted with respect to the collector or the brushes with regard to the magnetic field, then this horizontal component is no longer inactive, so that a current can be produced, at will in either direction (fig. 8 and 9).

The wryness in the field, which this specimen showed, can easily occur, when the centering of the armature between the poles of the magnet resp. the poleshoes, as well sideways as in the direction of the axis, is deficient, which can very well be the case in an air gap of not more than some tenths of a millimeter. This explanation requires, as was confirmed by the experiment, that, when the relative position of collector and coil is the right one, such

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1) We may neglect the armature reaction because it was very small, given the weakness of the currents. Besides, by taking into account this reaction hardly anything changes in the reasoning.

an unsymmetrical distribution of the magnetic force does not cause any error in the determinations.

Some difficulty arose in finding the right form and the right material, that allows the current to be taken off from the collector

Fig 6

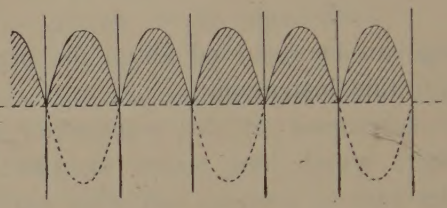


Fig 7

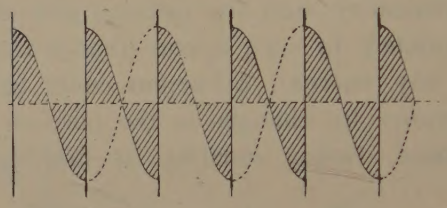


Fig 8

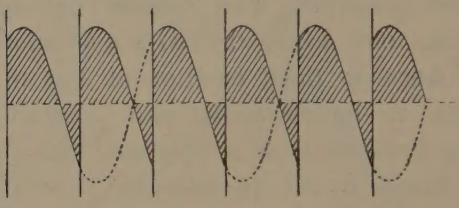
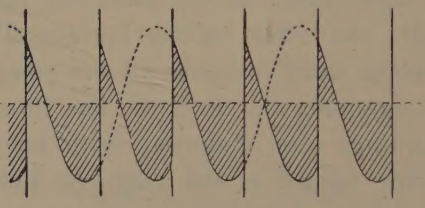


Fig 9



uninterruptedly during a long time. In one respect the conditions are not very exigent: fluxions in the strength of the current by changes of the resistance at the collector up to one percent can be allowed in practice without any reserve. On the other hand the reliability of the working must be guaranteed at a minimum of care and looking after. By making the diameter of the collector not to exceed some millimeters and by taking silver both for the collector and for the collector springs a method of transmission of the current was arrived at which gave excellent results during a long time at a stretch, without any greasing or oiling.

A freely suspended magnet cannot be prevented from executing slight oscillations in consequence of incidental accelerations and

retardations in the air, nor the frame which carries the axis of rotation from being set swinging around its pivots. To avoid an inconvenient agility of the hand under influence of such oscillations, care was taken to secure a slow indication of the currentmeter, which was obtained by making the instrument over-deadbeat by damping, the oscillations of the magnet being made as short as possible, besides which a damping by mechanical friction was applied for so far as coexistent with the requirements of an accurate working.

If the instrument was adjusted sideways on the aeroplane out of the "propeller-wind" about halfway between the two decks of a biplane an ascension was indicated when the machine flew at level keel, while an ascending was strongly exaggerated and a descent was not at all or hardly indicated. This shift in the direction of an ascension in the indications, which was observed in a higher or lower degree in a number of places on the aeroplane is caused by air currents generated by reaction from the aeroplane in its quick motion through the air. When fixed at the end of a wing at some distance sideways, the instrument gave undisturbed indications. In this position however the troubles of the oscillations mentioned above were felt more strongly.

It would be a valuable result if by means of this instrument a systematical investigation could be made of the vertical velocities of those reaction currents produced in the neighbourhood of an aeroplane in flight.

It is highly to be regretted that there is no possibility in our country of calibrating the scansimeterscale in an artificial air current. When the electrical resistances, the strength of the magnetic field and the sensitiveness of the currentmeter are known, this calibration consists in the research concerning the number of revolutions the screw makes at various velocities relatively to the air. Since the air-tunnel set up on a small scale by the "Kon. Nederl. Vereeniging voor Luchtvaart" could no longer be used, such a testing cannot be executed in our country.

Physiology. — *"Is there any Relation between the Capacity of Odorous Substances of Absorbing Radiant Heat and their Smell-Intensity?"* By Dr. G. GRIJNS.

(Communicated in the meeting of September 29, 1918).

For a thorough knowledge of the apperception of our senses we must first of all find out the special character of the stimulus to the receiving of which the organ is adapted. The study of light and sound teaches us that the quality is determined by wavelength, and its intensity by amplitude. With regard to smell and taste we do not know as yet what property of the odorous or the gustable substance determines the sensations they arouse.

Many attempts have been made to establish a relation between smell and certain qualities. Up to the present without success.

The chemical properties which, indeed, were thought of first and foremost, appeared to yield no satisfactory interpretation of all the peculiarities. ZWAARDEMAKER [1]), therefore, reaches the conclusion in his "Physiologie des Geruchs", that no direct relation exists between the chemical constitution of a substance and its smell.

LIÉGEOIS [2]) pointed out that a large number of odorous substances, when put on the surface of clean water, present the same phenomenon as PRÉVOST [3]) described for camphor and even calls it the "odoroscopic phenomenon". However, it turned out that a number of inodorous substances also present this phenomenon. So there is no argument for correlating smell with surface-tension phenomena, to which VAN MENSBRUGGE attributes the behaviour of camphor on water.

ZWAARDEMAKER's finding [4]) that a large number of odorous substances evolve an electrically charged vapour, when sprayed in aqueous solution, gave rise to the supposition, that electrical charges come

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[1]) H. ZWAARDEMAKER, Die Physiologie des Geruchs. Leipzig 1895.

[2]) LIÉGEOIS: Sur les mouvements de certains corps organiques à la surface de l'eau. Arch. de Physiol. 1868 T. I p. 35.

[3]) PRÉVOST: Annales de Chimie et de physique XXI p. 254, XXIV p 31, XL p. 1.

[4]) H. ZWAARDEMAKER, These Proc. XIX p. 44, 334 and 551.

into play here. ZEEHUIZEN [1]), however, has detected that also saponins and antipyretica, which do not smell, produce an electrical charge. This, therefore, is not a property peculiar to odorous substances alone.

TYNDALL [2]) previously observed that several odorous substances possess a great capacity of absorbing obscure rays. We might, therefore, expect a correlation between these two properties, as it is quite possible that odorous substances derive energy from their surroundings to transmit it to the olfactory organ.

If so, we must expect smell-intensity and absorbing capacity in different odorous substances to evince, if not proportionality, at all events parallelism.

On Prof. ZWAARDEMAKER's suggestion I undertook an investigation in this direction.

Since I purposed merely a preliminary orientation with regard to this problem, there was no need of being very accurate, so that many difficulties could be avoided.

I determined the absorptive capacity of dry air that had passed through a slightly curved tube filled with the fluid under examination and compared it with the capacity of pure dry air by a differential method.

On either side of a Nernstlamp a glass tube was placed closed at both ends with a plate of rocksalt. Behind these tubes were arranged MOLL's thermo-piles connected oppositely with a MOLL galvanometer. Air or gas could be sucked through one of the tubes by means of a spirometer. In front of the other tube an irisdiaphragm was mounted, the area of which could be read from a large protractor. The width of the opening, corresponding to every scale-division, was determined by measurement.

The deviation of the galvanometer was measured by the movement of the reflected image of an illumined copperwire on a millimeter-scale. For every determination dry air was first passed through the one tube, and then the diaphragm adjusted so as to make the galvanometer point to zero. This could readily be done, as the apertures in double screens, interposed between the lamp and the measuring tubes, could be covered or uncovered at the same moment.

Subsequently air was sucked for about ten minutes through the fluid under examination, and through the measuring tube. After this the diaphraghm was so adjusted that again removal of the

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[1]) These Proc. XX p. 1272.

[2]) J. TYNDALL: Heat as a mode of motion p. 366.

screens did not cause a deflection of the galvanometer. Next the tube was rinsed for a quarter of an hour, or longer if need be, with dry air and the zero-point determined again. If the two zero-points are approximately equal and the determination with the odour-containing air deviates, the ratio of the diaphragm apertures is that of the quantities of light transmitted, from which the absorption can be calculated.

In the same way dry air, sucked through the fluid under examination was led through a glass vessel provided with an opening, which could be closed by a glass plate, or to which ZWAARDEMAKER's diaphragm odorimeter could be adjusted.

By means of this apparatus, a description of which will soon be published, odour-containing air can be diluted to a known proportion. Into a cylindrical smell-chamber, from which the content is inhaled through a glass tube, the odorous air enters through an irisdiaphragm at the bottom, while free air can enter through lateral adjustable slits.

We estimate the dilution in which the scent almost fails to be recognized. So the absorptive capacity and the smell-intensity of a given mixture of odorous substance and air were known. We now calculated for every examined mixture how many times it had to be diluted to absorb just 1% of the incident light, and from this how many olfacts such a dilution would contain.

The subjoined table shows the results thus obtained:

Substance examined	Number of olfacts	Substance examined	Number of olfacts
Methylic alcohol	12	Bromoform	16°
Ethylic alcohol	7.2	Benzol	72
Ether	> 13°° [1]	Toluol	16°
Amylic acetate	20°°	Xylol	94
Glacial acet. ac.	36°	Toluidin	51
Propionic acid	22°	Eucalyptol	> 34°° [2]
Chloroform	14°	Eugenol	> 14°° [2]

If smell-intensity were related to capacity of absorbing radiant heat the above values might be expected to be approximately equal. Since they are so widely different such a relation must be denied.

[1] The dilution of ether-containing air was beyond the capacity of my measuring instrument.

[2] In both these cases the absorption was less than $^1/_2$ perc., consequently so minute as to elude measurement with my equipment.

Botany. — "*On the course of the formation of diastase by Aspergillus niger*", by Prof. F. A. F. C. WENT.

(Communicated in the meeting of September 29, 1918).

It is already a long time that I have intended to carry out an investigation on the influence which external conditions exercise on the formation of enzymes, more especially in *Aspergillus niger*.

In a certain sense this would be a continuation of an earlier investigation, which however related to another fungus, *Monilia sitophila* [1]). That my attention has now been turned to *Aspergillus niger* has its explanation in the fact that KATZ has expressed opinions about the formation of enzymes in this fungus, which seem to me hardly probable, but with regard to which a sure judgment can of course only be formed after a renewed investigation [2]). Other work has prevented me from carrying out my intention, and seems likely to do so in the near future; moreover in the meantime another investigator, HARALD KYLIN has repeated KATZ' investigation by a better method [3]). It might thus appear that there is no reason for publishing a paper, if my preliminary experiments had not given some results, which seem to me of sufficient importance to warrant publication even though the investigation is incomplete.

This preliminary inquiry dealt with the question how the formation of diastase by *Aspergillus niger* depends on its age. It seemed to me that in all comparative investigations insufficient attention has been given to this important point, namely the age of the cultures. Generally cultures of differing nutrition were all after the same length of development compared with one another, it being however evident, that the comparison was here made between unlike things, because the nutritive values of the solutions differed widely. It seemed to me that it would be necessary to trace the whole course of the formation of enzymes in successive intervals of time in each nutrient medium.

I have not extended the investigation to this length, but have

1) F. A. F. C. WENT, Ueber den Einflusz der Nahrung auf die Enzymbildung durch Monilia sitophila (Mont.) Sacc. Jahrb. f. wiss. Botanik. 36 1901.

2) J. KATZ, Die regulatorische Bildung von Diastase durch Pilze. Jahrb. f. wiss. Botanik. 31 1898.

3) HARALD KYLIN, Ueber Enzymbildung und Enzymregulation bei einigen Schimmelpilzen. Jahrb. f. wiss. Botanik. 53 1914.

only endeavoured to solve the problem in one definite case, how the formation of diastase takes place. For the culture-fluid a solution was always used containing 5 % glucose, 0.5 % NH_4NO_3, 0.1 % K_2HPO_4 and 0.05 % $MgSO_4$. This solution was sterilised by heating it on three successive days for half-an-hour to 100° C.

The inoculation was carried out by means of a platinum loop with fluid containing the conidia of *Aspergillus*. It is known that the latter remain floating on the surface of the fluid and that one cannot keep them submerged. One can take care to get a fairly uniform distribution on the surface of the fluid and then always inoculate the same amount with a loop. Thus one does not indeed obtain a complete uniformity of the number of conidia in the different culture flasks, but the differences are so small as to exercise no influence on the final result, at most a slight difference in development is observable in the first two days. It should be possible to ensure a greater uniformity of inoculation-material, but this would be fraught with so many difficulties that it is not worth while in view of the very small advantage it would yield. It needs no demonstration that the sowing of a single conidium is here wholly impermissible, because then one would have to reckon with great individual differences. These differences can only be compensated for by inoculation with a large number af conidia.

The fungi were grown in glass (Erlenmeyer) flasks and for comparative experiments the same quantity of culture-fluid was always placed in similar flasks. The latter were kept in a room at a constant temperature of 24° C. with variations of 0.5°.

The cultures were in darkness; artificial light only was used for inoculation and observation.

At first daily and later after two, three and more days determinations were made of the amount of enzyme present in the culture-fluid and in the fungus-mass of one of the flasks, but generally this determination was made for 2 or 3 flasks. Then at the same time the fungus-mass formed in one of the flasks was collected on a tared filter and weighed after it had been dried, so that an idea was obtained of the quantity of dry material which had been formed. The other fungus-mass was ground fine in a mortar with the help of a little kieselguhr, and afterwards extracted for one hour with the culture-fluid and subsequently this fluid was filtered off and examined with regard to enzyme: now if we know the quantity of enzyme present in the culture-fluid, which had therefore diffused outwards, then we have only to subtract this from the quantity found in order to ascertain how much enzyme was present in the mycelium.

The quantity of diastase was determined by finding how long it was before the starch had completely disappeared out of a starch-solution of definite strength, which had been mixed with the enzyme-solution, whence the quantity of enzyme must be inversely proportional to the time. The presence or absence of starch was investigated by the aid of an iodine-solution of known strength. Theoretically this is not the best method; it would be better to determine the quantity of sugar which is formed in a definite short time from the starch and then to consider the quantities of diastase proportional to the figures thus found. But this is impossible in the case of the diastase of *Aspergillus niger*, because there is here often such a very small quantity of enzyme, that the sugar could not be determined. For this reason it was thought preferable to divide the enzyme-solution to be examined, into a number of equal parts, and to mix each of these with the same small quantities of very dilute starch-solution and successively to test the mixtures after a definite interval for the presence or absence of starch with the aid of the same very dilute iodine-solution.

Preliminary experiments showed that when a solution is used which contains in 100 gr. water 62.5 mg. iodine and 62.5 mg. potassium iodide, 1 ccm. of this is enough to colour a starch-solution distinctly blue, when this contains 1 mg. soluble starch in 10 c.c. of water, whilst when this amount is only 0.2 mg., a definite red-violet colour still results.

There are formed from starch, before it is completely hydrolysed by the action of the diastatic enzyme of *Aspergillus niger*, erythro-dextrin-like intermediate products, which with iodine are coloured red; so long as such bodies were present it was assumed that the reaction was not ended; it was only when after mixing with the iodine solution, the colour remained yellow, that the starch was considered to have disappeared completely. Naturally all the fluids were under the same condition. For this reason 5 c.c. of the fluid to be investigated was always mixed in test-tubes with 5 c.c. of a solution which contained 0.08 pc. of soluble starch and with some drops of toluene; the latter was for the purpose of preventing any development of fungi or bacteria in the experiments of longer duration. The tubes were then shaken for a short time; and then placed in the dark at a constant temperature of 24° C. After a certain time the fluid was filtered off from one of the test-tubes and 1 cc. of the above mentioned solution of iodine was then added. If the colour was blue or red, this process was repeated some time later, until the colour was decidedly yellow.

The question was whether in this way, by repeated trials, trustworthy results could be obtained. In order to ascertain this, some preliminary experiments were carried out. From a culture of *Aspergillus niger* which was 5 days old, the fluid was filtered off and divided into two equal portions. One half remained unchanged, the other was again divided into two, and one of the halves was boiled for a short time to ensure destruction of the enzyme, and after cooling it was mixed with the unboiled portion, so that there were two fluids of which one must contain double the quantity of the enzyme in the other. 5 cc. of each of the fluids was mixed in a test-tube with 5 c.c. iodine-solution and after certain intervals each tube was tested for the presence of starch. It was found that the undiluted solution after 95 minutes was still coloured somewhat orange yellow by iodine, whilst after 100 minutes the colour was pure yellow, so that the colour-change had taken place after 97.5 minutes. In the twice-diluted solution the change had taken place in the interval between 180 and 190 minutes, therefore in about 185 minutes. The proportion of 97.5 : 185 is 1 : 1.9, whilst the quantities of enzyme were to one another as 2 : 1, so that the error in determination was not more than 5 %.

In another case the starch was hydrolised by the undiluted solution in 495 minutes, by the twice-diluted in 907 minutes and by the 10 times diluted in 4365 minutes. These figures are as 1 : 1.8 : 8.8 instead of as 1. : 2 : 10.

Lastly there was a third case in which the enzyme-solution had been diluted 2, 4, 8, and 16 times. The result is shown in the following table.

	Still coloured red with I.	Disappearance of starch	Average	Proportion
Undiluted solution	After 1 h. 45 min.	After 2 hours	1 h. 52 min.	1.—
Twice diluted "	" 3 " 20 "	" 4 "	3 " 40 "	2.—
4 times " "	" 7 " 30 "	" 8 "	7 " 45 "	4.1
8 " " "	" 13 " 30 "	" 16 "	14 " 45 "	7.9
16 " " "	" 27 " 00 "	" 28 "	27 " 30 "	14.7

When it is remembered how tentative these estimations were then the agreement between the determination and reality must be considered very satisfactory; the difference here amounts to much less than 10 % in most cases.

One can therefore conclude that in this way an idea can be obtained of the quantity of enzyme which is present in certain fluids. This only holds good on condition that these fluids have for the rest quite the same composition. Now it has unfortunately to be stated that the culture-fluid of each fungus changes in the course of the development, partly because certain bodies from the fluid are taken up by the developing mycelium, partly also in consequence of secretions by the fungus.

In this respect therefore the cultures of *Aspergillus* at different stages are not quite comparable. The concentration of the H-ions could be made equal, by the addition of acid or alkali, but that would not completely meet the case, because other bodies may certainly be present which hasten or retard the reaction and which, at least at the present time, cannot be determined.

There is indeed another method conceivable which would consist in mixing culture-fluids of different stages with one another, after part of them had been boiled to destroy the enzyme. By this means one would then be able to trace whether in a given solution substances were present which hasten or retard the enzyme-action. From some preparatory experiments it appears that something may perhaps be obtained by such a method. So, for example, the fungus from a culture 7 days old, was finely ground up, and then extracted with its own culture-fluid, and the solution was then diluted with an equal volume of *a.* its own culture-fluid, after this had been boiled to destroy its enzyme, *b.* a culture-fluid similarly boiled from a culture which was 17 days old, and *c.* a like solution of a culture which was 3 days old. On investigation it was found that when 5 c.c. of the above mentioned solutions were mixed with 5 cc. of a solution of soluble starch of 0.08%, 1320 minutes elapsed in the cases of *a* and *b* before all the starch had disappeared, whilst in the case of *c*, this period amounted to 900 minutes. Hence there was in *c* either present an accelerator, or the solutions *a* and *b* contained retarding substances which were wanting in *c*. For the rest this method was not worked out further on account of the condition mentioned at the beginning of this paper, and the figures obtained must therefore be received with a certain reserve. It will, however, be seen that they nevertheless give some idea of the course of the formation of diastase in *Aspergillus niger*.

Before I further mention the results of the investigation I will first indicate by means of an example, exactly how the figures were obtained. Three flasks, each containing 75 c.c. of culture-fluid were investigated, after a culture of *Aspergillus niger* had been in them

for 3 successive days. From flask 1 the culture-fluid was filtered off, the fungus-mass ground fine in a mortar with kieselguhr, and afterwards extracted with the culture-fluid. Subsequently it was filtered and the filtrate distributed in test-tubes, so that each tube got 5 c.c. of fluid; then to each was added 5 c.c. of a 0.08 % solution of soluble starch. After 60 minutes the fluid of one of the tubes was examined in the manner already described with iodine; the colour was blue. After 180 minutes the second tube gave a pale claret-red, after 225 minutes the colour of a third tube was light-yellow. It may therefore be assumed that the hydrolysis of the starch had taken place in about 200 minutes (1).

A second flask was treated in the same way; the reactions with the iodine solution can be seen in the following table:

After	60	minutes.	blue
„	180	„	reddish-violet
„	220	„	„ „
„	255	„	pale claret-red
„	295	„	reddish-yellow
„	340	„	yellow

so that the time taken for hydrolysis was about 315 minutes (2).

The fungus-mass in a third flask was collected on a tared filter, then washed and dried; the dry material weighed 69.5 mg. The culture-fluid alone was examined for enzymes by the method described for the other flasks. The following table shows what reactions were obtained with the iodine-solution.

After	175	minutes	blue
„	250	„	blue
„	335	„	bluish-violet
„	485	„	reddish-violet
„	1055	„	reddish-yellow
„	1130	„	pale reddish-yellow
„	1190	„	yellow

It may therefore be concluded, that the hydrolysis of the starch required about 1160 minutes (3).

If the quantity of enzyme which is necessary to hydrolyse the soluble starch in 10 c.c. of a 0.04 % solution in 100 minutes is put equal to 10 then the quantity in the test-tubes in case (1) is $\frac{100}{200} \times 10$, in (2) $\frac{100}{315} \times 10$, in (3) $\frac{100}{1160} \times 10$, therefore the whole quantity in the 75 cm. is $\frac{75}{5}$ or 15 times greater.

Therefore the calculation in the case of (1) gives a quantity of diastase of 75,0, in (2) 47.6 and in (3) 12.9. If it is assumed, that in flasks 1 and 2 the same quantity of enzyme was present in the culture-fluid as in flask 3, then there remains for the enzyme in the mycelium for (1) 62, and for (2) 34.7. These figures appear to diverge widely, but nevertheless we shall soon show, that there is a great regularity in the quantity of enzyme found on daily investigation.

The experiment which is here described in detail was the beginning of a whole series of determinations. Altogether 42 similar flasks were inoculated with *Aspergillus*, and after a given number of days three of them were each time examined in the manner described. The result is given in the appended table. In column I the number of days is given, during which the culture lasted, in column II the quantity of enzyme in the mycelium and culture-fluid together, in column III the same for the other flask, in column IV the mean of II and III, in column V the amount of enzyme in the culture-fluid alone of the third flask, in column VI the difference between IV and V, and finally in column VII the dry weight of the crop of fungus in flask 3. Wherever the sign $<$ is found in the table, this indicates that the last test-tube still gave

I	II	III	IV	V	VI	VII
3	75.0	47.6	61.3	12.9	48.4	69.5 mgr.
5	44.1	44.1	44.1	4.2	39.9	319.0 „
7	66.6	41.8	54.2	3.7	50.5	398.0 „
8	21.7	30.3	26.0	3.0	23.0	538.5 „
10	< 6.0	< 6.0	< 6.0	2.3	< 3.7	549.0 „
12	4.1	7.5	5.8	1.8	4.0	699.0 „
15	< 2.8	< 2.8	< 2.8	trace	< 2.8	939.0 „
17	2.3	3.3	2.8	trace	2.8	936.5 „
19	< 1.6	< 1.6	< 1.6	0.5	< 1.1	943.5 „
23	1.0	2.6	1.8	2.0	0.0	899.0 „
26	6.2	10.0	8.1	2.5	5.6	709.0 „
30	16.7	4.5	10.6	3.3	7.3	782.0 „
33	11.1	29.3	20.2	16.7	3.5	742.5 „
37	16.6	50.0	33.3	11.1	22.2	734.5 „

a reaction with the iodine-solution, so that the limit of hydrolysis had not yet been reached. Wherever "trace" is written, the amount is less than 0.5, although a change in reaction was still clearly observable.

The figures of columns II and III appear at first sight to diverge widely, but on nearer inspection a certain regular course is however noticeable, by which column IV is justified. The amount of enzyme in the culture-fluid is generally very small, so that the course of column VI is scarcely different from that of column IV. There are a few exceptions to this, but I think it better to give the complete observations first and then only to begin a discussion of the figures found.

A second series of figures had reference to a number of cultures which had been made at the same time as those already described, and which only differed from them in that instead of 75, 150 cc. of culture-fluid was used and in larger flasks, in consequence of which the surface of the culture-fluid amounted to 47 square cm. instead of 24 as in the foregoing series. There was indeed the possibility that on this account the development of *Aspergillus* might continue longer,

	Enzyme in fungus and culture-fluid	Enzyme in culture-fluid only
after 3 days	65.2	6.7
„ 5 „	75.0	5.4
„ 7 „	75.0	3.6
„ 8 „	75.0	< 3.0
„ 10 „	30.0	2.2
„ 12 „	30.0	< 1.9
„ 15 „	13.6	1.5
„ 17 „	11.0	1.2
„ 19 „	1.7	0.0
„ 23 „	4.0	0.0
„ 26 „	4.2	0.0
„ 30 „	11.3	0.0
„ 33 „	44.1	0.0
„ 37 „	4.5	0.0

and with it also the formation of enzyme. Of these flasks half were directly examined as to enzyme, the others after they had stood for an hour with the finely ground fungus-mass and had then been filtered. The results obtained are shown in the fore-going table.

In order also to trace during a longer period the course of diastase formation, a new experiment was begun with 99 flasks each containing 300 c.c. of nutrient solution and treated exactly in the same way as the first series with this one difference, that this time account was taken of the quantity of liquid which remained when cultivation was over. In consequence of consumption by the fungus and of evaporation this quantity diminishes, a fact that in the above experiment of 37 days could not have so strong an influence on the results as in those lasting no less than 149 days. In consequence of this the calculation was more complicated, but still sufficiently simple, and after what has already been said, it will not be necessary to give the slightly modified calculation here; we may only remark that the figure of column VI is therefore not exactly the difference between IV and V. With this qualification the different columns in the following table, have the same significance as in the first table already given.

The figures of column IV are represented graphically in fig. 1. In this figure the age of the culture (in days) is measured on the axis of

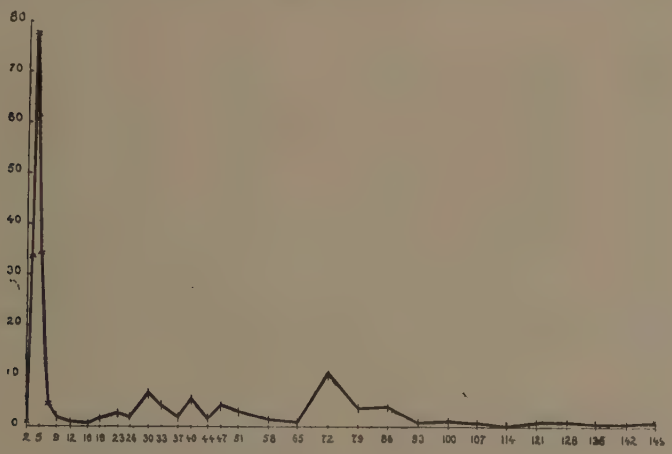

Fig. 1.

the abscissae, whilst the amount of diastase is respresented by the ordinates. In figure 2 the figures of column VI are represented in the same way. Valid objections may perhaps be raised against the latter figures, because in these cases the amount of enzyme was deducted, which occurred in the culture-fluid of a culture distinct

I	II	III	IV	V	VI	VII
2	0.75	0.75	0.75	0.41	0.33	1 mgr.
3	24.19	42.25	33.22	9.71	23.56	85 „
4	44.37	111.11	77.74	10.98	66.64	344 „
5	24.40	43.48	33.94	2.11	31.90	754 „
6	8.00	18.76	13.38	0.76	12.64	952 „
7	2.40	6.34	4.37	0.29	4.09	1236 „
9	1.50	1.50	1.50	0.05	1.45	1577 „
12	1.00	0.50	0.75	0.00	0.75	1562 „
16	0.50	0.50	0.50	0.00	0.50	1259 „
19	1.64	1.00	1.32	0.30	1.01	1117 „
23	1.11	4.45	2.78	0.76	2.04	1095 „
26	3.12	1.24	2.18	1.20	0.97	1080 „
30	4.44	8.70	6.57	4.41	2.11	1025 „
33	4.17	4.17	4.17	0.82	3.46	1073 „
37	1.64	1.50	1.57	1.11	0.57	1087 „
40	6.00	4.44	5.22	4.00	1.10	1031 „
44	1.26	1.12	1.19	1.37	0.00	1040 „
47	4.48	2.00	3.24	3.48	0.00	1034 „
51	1.25	3.13	2.19	7.32	0.00	1035 „
58	0.94	0.94	0.94	1.90	0.00	1039 „
65	1.12	0.72	0.92	0.98	0.00	987 „
72	0.50	20.00	10.25	4.60	5.68	982 „
79	6.40	0.50	3.45	4.56	0.00	1039 „
86	0.38	7.50	3.94	1.08	2.89	1009 „
93	0.04	0.42	0.23	1.71	0.00	966 „
100	0.30	0.42	0.36	0.20	0.16	956 „
107	0.10	0.36	0.23	1.56	0.00	992 „
114	0.10	0.10	0.10	0.08	0.01	933 „
121	0.08	0.44	0.26	0.08	0.17	973 „
128	0.08	0.68	0.38	0.06	0.32	958 „
135	0.00	0.12	0.06	0.57	0.00	917 „
142	0.08	0.00	0.04	0.05	0.00	962 „
149	0.05	0.05	0.05	0.00	0.05	922 „

from that in which the amount of enzyme in the mycelium was

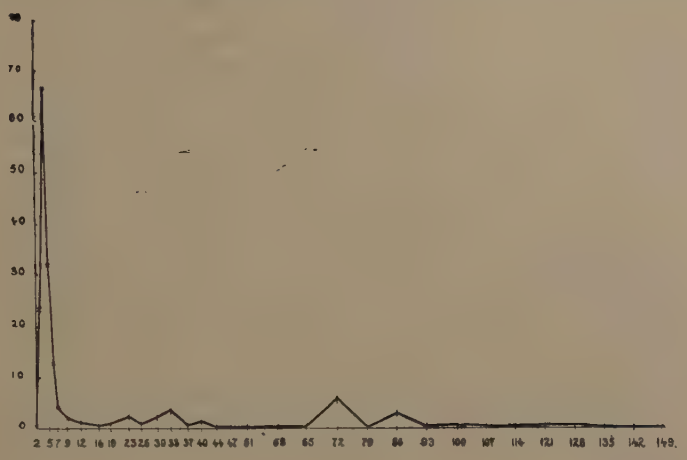

Fig. 2.

determined. To this must be ascribed the fact that such cases occur as that after 79 days, where in the fungus plus fluid an average of 3.45 enzyme occurs, but in the fluid alone 4.56, i. e. more. This is of course absurd, and if its own culture-fluid could have been used, such figures would not have resulted. Nevertheless I have included these figures and represented them in figure 2, in order to show the general course of the presence of diastase in the fungus-mass itself. It is evident that this curve is, generally speaking, exactly identical with that of figure 1. This is easily understood when it is seen from the figures of column V that in general the culture-fluid contains no very appreciable quantity of enzyme; it is only when much enzyme is present in the mycelium, that a quantity can be found in the surrounding fluid, which is not inconsiderable. How it is that the enzyme occurs there, must be left an open question. It might be thought to have arisen from cells already dead, but on the other hand a somewhat greater quantity is only found in very young mycelia.

The enzyme found in the culture-fluid can therefore be left without further consideration in order that attention may be concentrated on the diastase which is found within the mycelium of the fungus. From all the tables and also from figures 1 and 2 it is quite evident that after germination a considerable increase of enzyme is observable, which very quickly reaches a maximum and afterwards shows an almost equally rapid diminution. There will naturally be an inclination to consider this increase in relation to the development of the

mycelium, but the figures for the fungus-crop in the different tables show that this cannot be so. Moreover in figure 3 a representation

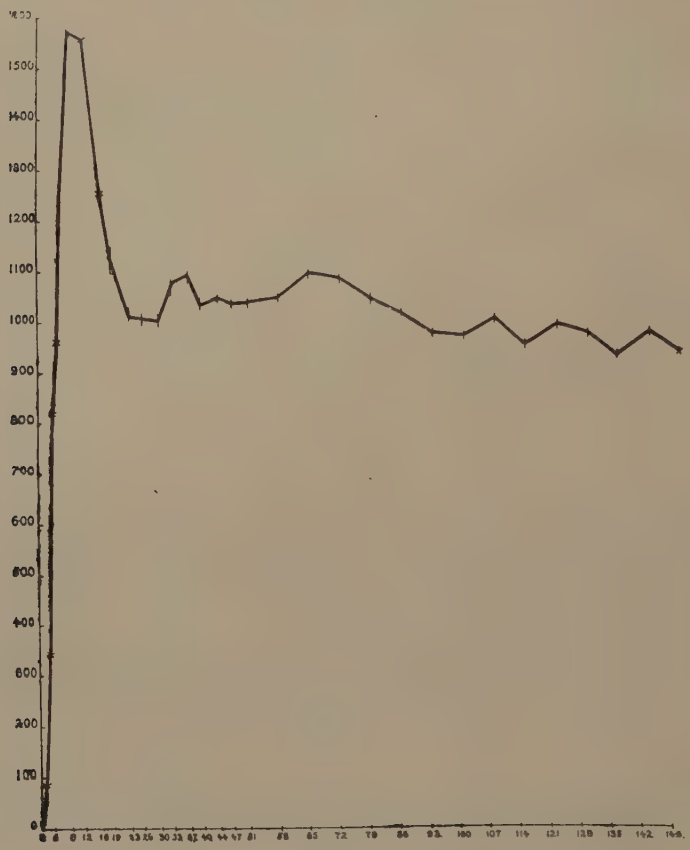

Fig. 3.

is given of the development of the mycelium in the last-mentioned experiment. There the duration of the development, given in days, is measured on the axis of the abscissae, whilst the ordinates give the dry-weight of the fungus-mass. In the first days a strong increase of the crop is evident, which is followed by a decrease, probably caused by the preponderance of the processes of dissimilation over the assimilation. On working up the mycelium for the enzyme a large portion of the older cultures was found to be dead, a phenomenon quite easy to establish, because the hyphae felt soft and flaccid and no longer elastic, as in the young cultures; but this could only with certainty be observed, when the fungi were some months old.

It is however quite clear that no proportion exists between the amount of dry matter of the fungus and the quantity of enzyme

formed. Moreover it is seen that there is indeed in the beginning of the development an increase in the dry matter as well as in diastase, but that the increase in enzyme comes to a standstill, whilst the fungus-mass continues to increase for many days.

Do the experiments described indeed afford a proof that the amount of diastase in the fungus increases during the first days and then again undergoes a diminution? May it not be that the culture-fluid, which is used for the extraction of the enzyme, undergoes a slow change of such a nature as to accumulate a substance which destroys the enzyme or at least opposes its action? Or conversely may not there be some accelerator which is present at the beginning and first increases, and later diminishes? A decided answer to such questions can only be given when the method has been further worked out which was briefly described at the beginning of this paper. But nevertheless there are facts which make the explanations suggested here very improbable. I therefore point out that in all series of experiments the decrease in the amount of diastase does not take place regularly, but that later again a sudden increase is observed. It is not readily conceivable that the composition of the culture-fluid should suddenly undergo such a change that the sudden change in the figures could be thus explained. One can only think of an increase of the amount of diastase within the cells of the fungus. And when in this case no great influence on the figures for the diastase can be assigned to the culture-fluid, then this cannot be assumed in the other cases either. It may therefore be expected with fairly great certainty that the general course of the curve of figure 1 or 2 gives a picture of the actual quantities of diastase which occur in the mycelium of *Aspergillus niger*.

If we may accept the above conclusion as correct, then there follows more from it. Firstly, if we disregard for a moment the later irregularities of the curve a very quick initial rise is observable, followed by an almost equally strong fall. The former would be explained by the constant formation of more fresh enzyme, but the fall? There hardly remains anything but the conclusion that destruction of enzyme is always going on within the cell; this destruction then in the first days is compensated for and exceeded by the new formation, which latter very quickly comes entirely to a standstill or becomes so slight that it is far from being sufficient to keep the quantity of enzyme at a level.

I now return to the irregularities which are to be seen in the falling part of the curve. It is seen from the first two tables that after about one month the amount of diastase in the fungus suddenly

shows a considerable increase. These experiments did not last long enough to show what the further course would be, and for this very reason the last series of flasks was started, in order to trace for a longer time the behaviour of *Aspergillus niger* in this respect. It was found that here also after about one month the rise of the enzyme-content occurred; thereupon there followed again a decrease and then again a rise, whilst after about three months the decrease became permanent, so that finally practically no diastase remained. The oscillations of this curve are explicable by considering that the great rise in the first 3—4 days coincides with a vigorous development of the young mycelium. The same takes place again later when the fungus has formed new conidia and these, having been shed into the culture-fluid, germinate there; each time that this happens a sudden increase in the quantity of the diastase will be observable. It is self-evident that it cannot always be predicted at what time in the course of the culture this will occur, but we may safely argue that the culture-fluid gradually will become less and less suited for the germination of the conidia, and that therefore this phenomenon will gradually stop altogether. If this explanation is correct it would therefore be possible to make the formation of diastase in *Aspergillus niger* go on for a much greater length of time by renewing the nutrient solution.

The phenomena which have just been described are thus of a secondary character and all study of the formation of enzyme in *Aspergillus niger* must be limited to the first stages of development. One cannot indeed say whether the course of the formation of enzyme would be the same with different nutrition, but this may however be expected. It is therefore not only impermissible to draw conclusions from the quantities of enzyme which occur in cultures which are e.g. some weeks old, but one must try to follow accurately the course of the enzyme-formation during the first days of development of the fungus in the case of each kind of nutrition; then only can conclusions be drawn respecting the influence of a given nutrition on the enzyme-formation of *Aspergillus niger*.

It is evident that these conclusions only concern the formation of diastase in *Aspergillus niger*; but they will nevertheless oblige one to be careful about conclusions as to other enzymes and other fungi; in further investigations the possibility must be borne in mind of similar results occurring.

Summing up, it is found that in *Aspergillus niger* during the first days after germination a great quantity of diastase is formed in the

mycelium (on feeding with glucose as source of carbon and NH_4NO_3 as source of nitrogen), and that in addition, destruction of this enzyme takes place, which at first becomes negligible in comparison with the formation, but which soon makes itself so evident that the total quantity quickly decreases, after it has reached a maximum about 5 days from the commencement of germination. Into the nutrient fluid there passes never more than a very small part of the total quantity of enzyme, occurring in the mycelium; this is perhaps partly derived from dead cells.

Utrecht, August 1918.

Physics. — "*Investigation by means of X-rays of the crystal-structure of white and grey tin.* II. *The structure of white tin.*" Communication N°. 2*a* from the Laboratory of Physics and Physical Chemistry of the Veterinary College at Utrecht, by A. J. Bijl and N. H. Kolkmeijer. (Communicated on behalf of Prof. W. H. Keesom, Director of the Laboratory, by Prof. H. Kamerlingh Onnes).

(Communicated in the meeting of September 28, 1918).

In Communication N°. 1 (June 1918) we communicated, that we had taken Röntgenograms of white and grey tin by the method of Debije and Scherrer and we gave a description of some particulars about the arrangement of these measurements. Moreover we showed in a drawing for both states of tin mentioned the places of the interference-maxima in a plane, perpendicular to the axis of the bars, indicating at the same time the intensities of the interference-lines. The photo of the grey tin indicated, that this material is crystalline, and on comparing the photos for the two states, it was evident, that it possesses a crystal-structure, which differs from that of white tin. We have now determined from those photos the crystal-structure as well of white as of grey tin, and shall communicate in this paper our results for white tin, reserving those for grey tin for communication N°. 2*b* (these Proceedings).

In table I the intensities of the interference-lines are inserted in the first column: vf means very feeble, f feeble, m moderate, s strong and vs very strong.

In the second column are inserted, expressed in tenths of mm, the distances of the intersections of the interference-lines with a plane through the axis of the incident Röntgen-beam and perpendicular to the axis of the preparation, to the point where that first axis, prolonged, would meet the film, measured on the film when developed on a plane. In the third column the values of $sin^2 {}^1/_2\theta$ are ranged, as computed from the above-mentioned distances and the data of the apparatus, which were given in Comm. N°. 1.

Now the values, given in the third column must undergo a

TABLE I.

Intensity	Half of diameter of interf.-line	$\sin^2 \frac{1}{2}\theta$	$\sin^2 \frac{1}{2}\theta$ corrected	corr. $\sin^2 \frac{1}{2}\theta \times \frac{1}{0.01734}$	smoothed	Index-triples with $h_3 = 0$	Index-triples with $h_3 = 1$	Index-triples with $h_3 = 2$	Index-triples with $h_3 = 3$
1	2	3	4	5	6	7	8	9	10
m	168	0.0924	0.0790	4.6	4.5	2.0.0.			
vf	208	0.1393	0.1243	7.2	7		1.0.1.		
vs	228[5]	0.1663[5]	0.1502	8.7	9	3.0.0. 2.2.0.	1.1.1.		
f	280	0.2424	0.2253	13.0	13	3.2.0.			
s	311	0.2928	0.2762	15.9	16	4.0.0.	3.1.1.		
f	322	0.3114	0.2945	17.0	17	4.1.0.			
vf	349[5]	0.3591	0.3433	19.8	20	4.2.0.			
m	362	0.3813	0.3653	21.1	21				
vs	390	0.4319	0.4173	24.1	24		3.3.1.	0.0.2.	
vf	416	0.4794	0.4666	26.9	27			2.0.2.	
m	438[5]	0.5208	0.5079	29.3	29.5				
s	464	0.5675	0.5565	32.1	32	4.4.0.	5.1.1.	2.2.2.	
s	500[5]	0.6332	0.6243	36.0	36	6.0.0.			
vs	538[5]	0.6990	0.6922	39.9	40	6.2.0.	5.3.1.	4.0.2.	
vf	554	0.7248	0.7182	41.4	41.5				
s	586[5]	0.7687	0.7640[5]	44.1	44			4.2.2.	
vf	597	0.7923	0.7878	45.4	45.5				
vs	627	0.8352	0.8325	48.0	48				
f	640	0.8525	0.8499	49.0	49	7.0.0.		4.3.2. 5.0.2.	
m	690	0.9114	0.9104	52.5	52.5	6.4.0.			
f	721[5]	0.9415	0.9406	54.2	54				0.0.3.
m	757[5]	0.9686	0.9686	55.9	56		5.5.1. 7.1.1.	4.4.2.	1.1.3.
vf	790[5]	0.9863	0.9863	56.9	57				

correction [1]) in connection with the thickness of the preparation. This correction is not needed, when the absorption of the characteristic rays in the preparation can be neglected, which was the case for many of the materials, which were till now examined by the method of DEBIJE and SCHERRER. This does not hold however in our case (*Cu*-radiation), in which the radiation penetrated only to a very small depth, practically not at all in the preparation. The corrected values of $sin^2 {}^1/_2 \theta$ are found in the fourth column of the table.

During the computations for the drawing up of a formula for $sin^2 {}^1/_2 \theta$ it appeared that the majority of the values in the fourth column, in dividing them by 0,01734, give a quotient, which differs only little from a whole number. These quotients are found in the fifth column. The values of these quotients, smoothed to half a unity, are inserted in the sixth column.

MILLER [2]) has measured the crystal-form on crystals obtained by electrolysis. It appeared to belong to the tetragonal system with a proportion of axes $a : a : c = 1 : 1 : 0{,}3857$. So it lay at hand, to try to find agreement of the fourth column with a formula of the form:

$$sin^2 \tfrac{1}{2} \theta = A (h_1^2 + h_2^2) + B h_3^2 = A (h_1^2 + h_2^2 + q h_3^2),$$

in which $A = \frac{\lambda^2}{4a^2}$ and $B = Aq = \frac{\lambda^2}{4c^2}$. By a and c is meant now in this the edges of the elementary parallelepiped or of the elementary cell. It lies also very much at hand to choose for A the above-given value 0,01734, and so to seek further agreement of the numbers in the sixth column with an expression $h_1^2 + h_2^2 + q h_3^2$, in which evidently q was then to differ little from a whole number.

By choosing $q = 6$, a proper agreement appears to be obtained. In order to see this, it is advisable to compose a table of the values

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[1]) Comp. also P. DEBIJE, Phys. Z.S. 18 (1917) p. 5. For the value of this correction we found (to diminish the value of $sin^2 {}^1/_2 \theta$ with):

$$\frac{r}{4R} sin \theta \left\{ \frac{R}{d} + cos \theta + \sqrt{\frac{R^2}{d^2} + \frac{R}{d} cos \theta + 1} \right\}$$

Here r means the radius of the circular cross section of the bar, R that of the camera, d the distance of the centre of the camera to the opening of the screen, by which the rays enter. Second and higher powers of r/R are neglected. [Added during translation: The deduction of this formula has in the mean time been inserted in A. J. BIJL. Thesis for the Doctorate, Utrecht 1918 p. 22].

[2]) W. H. MILLER, Ann. d. Phys. u. Ch. 58 (1843), p. 660. See also H. VON FOULLON. Jahrb. d. Kais.-Kön. Geol. Reichsanst. Wien 34 (1884), p. 367.

of $h_1^2 + h_2^2$ for the different combinations of h_1 and h_2 ranged in the succession of increasing values of $h_1^2 + h_2^2$. On comparing this series of values of $h_1^2 + h_2^2$ with the numbers of the sixth column, the index-triples with $h_3 = 0$, given in the seventh column (not printed in italics), are found to satisfy. By adding 6 to all the values of $h_1^2 + h_2^2$ and comparing again with the numbers of the sixth column, in the same manner the index-triples with $h_3 = 1$ of the eighth column are found. In a similar manner the ninth and tenth columns are composed.

Apart from the first four lines, in connection with the deviations which exist for the small values of θ (comp. p. 498), of the remaining 19 lines, there are only 2, which do not belong to index-triples (not printed in italics), of which not all the indices are either even or odd. On considering whether all the lines to be expected, which fulfill this condition, are present, it appears that 5 of them still ought to appear. The index-triples meant here, are collected and printed in italics (columns 7—10) and assigned to those lines, of which the place is nearest to the place expected for these lines.

The difference between the places, where these lines are found and those, where they would be expected, is in all those cases minute, and can for two cases be ascribed to the fact, that the line meant, cannot be seen separated from a neighbouring line and in all cases does not exceed (in connection with the small value of A) the uncertainty of observation.

There is evidently indication enough for accepting a structure that is in accordance with the exclusive occurrence of interference-lines, which must be ascribed to reflections on net-planes with only even or only odd index-triples.

The absence of index-triples of which not all indices are either even or odd, points to centered sides.

From the atomic weight (119,0), the specific weight at 18° C. (7.285), AVOGADRO's number ($6{,}06.10^{23}$) and the wave-length of $Cu_{K\alpha}$ ($1{,}541{,}10^{-8}$) follows with $A = 0{,}01734$ and $q = 6{,}06$ [1]) (which value later on appeared still better to satisfy than 6), for the number of atoms per cell the value 3,02.

Now, we come to the number 3 by placing atoms in the corners of the cells and in the centra of the prism-faces.

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[1]) [Note added during translation]. By a somewhat different method of calculation A. J. BIJL, l.c. p. 33 obtained the value 5.988. This fact points to a greater probability of $q = 6$ accurately. In this case the projection of all the net-points on a diagonal-plane through the smallest edge of the elementary.cell would be a net of equilateral triangles with side $2{,}37 . 10^{-8}$ cm. (Comp. A. J. BIJL, l c. p. 34).

This is, so far as we know, the first example, in which atoms are found in the centres of the prism-faces, not however in those of the bases.

In excess of the above-mentioned triples, still other triples must in the case of this structure give occasion to interference-lines. In this case namely the structure-factor becomes $S = 1 + e^{\pi i(h_2+h_3)} + + e^{\pi i(h_3+h_1)}$. The value of this is 3, when all indices are either even or odd and in the other cases 1 or -1. So it is evident that planes the indices of which do not fulfill the condition mentioned, will be present as interference-lines, though, under equal circumstances for the rest, with $^1/_9$ only of the intensity of the other lines. So one cannot wonder that a few of this kind of planes can give sufficient intensity to the interference-lines and that their index-triples are present in colums 7—10 of table I.

In order to be able to judge about the agreement between the place and intensity of the interference-lines, which are to be expected on the basis of the given structure and the observed ones, a table has been drawn up, in which the values of $sin^2 \frac{1}{2}\theta$, as well for $Cu_{K\alpha}$-, as for $Cu_{K\beta}$-radiations, are collected, as computed for all the index-triples that come into consideration, with their relative intensities. Of this table II forms an extract in which are omitted in the first place all the triples for which the structure-factor not equals 3, and in which for the rest out of each group of index-triples which can be esteemed to produce lines which lie so near to each other that these together will give on our Röntgenogram only a single line, only that triple is noted that is the origin of the most intense line of that group. If such a group contains α- as well as β-lines, then the strongest only of both kinds is given. [1])

Deviations between computed and observed values of $sin^2 {}^1/_2\,\theta$, which are larger than corresponds to the degree of accuracy of the measurements, only occur in table II at the first three lines. About this it must be remarked that these lines are very hazy, which, in connection with the veiling of the film as a consequence of the action of white radiation in the Röntgenbeam makes the reading

[1]) During Comm. Nr. 1 being under press we learned, that Dr. SCHERRER and Prof. JOHNSEN had taken a Röntgenogram of white tin and Prof. JOHNSEN proposed to us, to exchange the observed values mutually. The results, then communicated to us cordially by Prof. JOHNSEN, give, especially at lower values, all of them somewhat smaller values of $\sin^2 {}^1/_2\,\theta$ than ours. Perhaps this might be ascribed to the correction applied by us for thickness of bar being somewhat too small. In the notes added to table II a few of the differences between the results of SCHERRER and JOHNSEN and ours are discussed.

TABLE II.

Intensity	$\sin^2\frac{\theta}{2}.10^3$ observed	$\sin^2\frac{\theta}{2}.10^3$ calculated for $Cu_{K\alpha}$	Relative intensity (calculated) of the α-lines	$h_1\ h_2\ h_3$	$\sin^2\frac{\theta}{2}.10^3$ calculated for $Cu_{K\beta}$	Relative intensity (calculated) of the β-lines	$h_1\ h_2\ h_3$
m	79 [1]	69	9.0	200			
vf	124				113	9.0	111
vs	150	140	9.0	111			
f	225				225	9.0	311
s	276	278	9.0	311	280	3.6	420
f	294[5]	feeble α-line					
vf	343	347	3.6	420	337	3.0	331
m	365 [2]				feeble β-lines		
vs	417 [3]	417	3.0	331			
vf	467				449	4.5	511
m	508 [4]	490	2.6	202	504	1.0	600
s	556[5]	556	4.5	511	561	3.6	531
s	624	624	1.0	600	620	3.3	422
vs	692	695	3.6	531			
vf	718				729	1.4	640
s	764	767	3.3	422			
vf	788				785	2.6	711
vs	832[5]				feeble β-lines		
f	850 [5]				844	1.2	602
m	910 [6]	902	1.4	640	897	2.25	731
					900	2.25	622
					904	2.25	313
f	941				953	1.1	820
m	969	972	2.6	711			
vf	986				feeble β-lines		

[1]) In SCHERRER and JOHNSEN's results this line is preceeded by f 57, which may be considered as 56 9 200 β. The values which S. and J. give for the first three lines, vs 72, m 116 and vs 143 namely, give better agreement with the places of the strongest lines, calculated, than ours.

[2]) Lines 343 and 365 are almost not to be separated on our film. They are given by S. and J. as one line, s 348.

[3]) S. and J. give 404[5], presumably by stronger forthcoming of 202 β (at 396) intensity 2,6.

[4]) s 491 of S. and J.

[5]) At the lines 832[5] and 850 the film shows almost uninterruptedly points from 819 to 866. S. and J. denote the intensities with s to m and f.

[6]) S. and J. have s 906.

somewhat uncertain, and that further the correction for thickness of preparation brings with it a relatively large uncertainty for these lines.

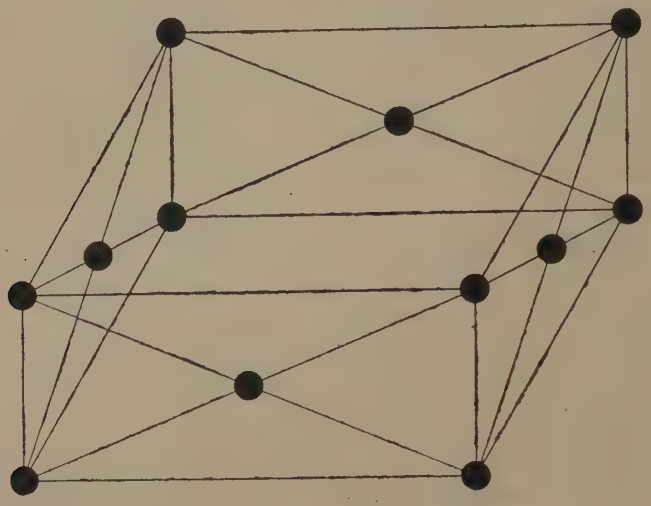

Fig. 1.

From the given values of A and q, there follows for the edges of the elementary-cell: $a = 5{,}84.10^{-8}$ and $c = 0{,}406\, a = 2{,}37.10^{-8}$ cm. In fig. 1 a representation of an elementary-cell is given.

In the space-netting, built of these cells, we see alternating equidistant layers with distance $1{,}19.10^{-8}$ cm. The first layer has a netting of squares with side $5{,}84.10^{-8}$ cm., the next a netting of squares with side $4{,}13.10^{-8}$ cm., which last squares are just above the squares symmetrically inscribed in that of the first layer.

The dense crowding of these planes indicates a strong force-exertion in a direction, perpendicular to the layers; perhaps the occurrence of needle-shaped combinations is connected with this fact.

Röntgenometrical investigations teach us to cast a look in the structure of the crystal and may for that reason lead us to a more rational choice for the system of crystallographic axes than the former crystallographic methods. The white tin procures an example of this. So e.g. the bipyramid, accepted by MILLER as (111) (proportion of axes $a : a : c = 1 : 1 : 0{,}3857$) would, conform to the system of axes formed from the edges of the elementary cell proposed by us, be indicated as (403); just so the planes (110), (100), (101) of MILLER by (100), (110), (223) resp. according to us.

Physics. — *"Investigation by means of X-rays of the crystal-structure of white and grey tin.* III. *The structure of grey tin"*. Communication N°. 2*b* from the Laboratory of Physics and Physical Chemistry of the Veterinary College at Utrecht, by A. J. BIJL and N. H. KOLKMEIJER. (Communicated on behalf of Prof. W. H. KEESOM, Director of the Laboratory, by Prof. H. KAMERLINGH ONNES).

(Communicated in the meeting of September 28, 1918).

In this communication we shall deduce the crystal-structure of grey tin from the Röntgenogram, taken by us and represented by a drawing in Communication N°. 1 (June 1918 p. 408) at the same time as that of white tin. With this aim in the first column of table I are given the distances of the intersections of the interference-lines with a plane, through the axis of the incident beam of X-rays and perpendicular to the lengthdirection of the preparation, to the point where this axis, prolonged, would meet the film, measured on the film when developed on a plane and corrected for thickness of preparation (Comp. Comm. N°. 2*a*, p. 496). The intensities are denoted by vf very feeble, f feeble, m moderate, s strong, and vs very strong. The second column contains the values of $sin^2 \frac{1}{2}\vartheta$ deduced therefrom, (radius of camera 27,3 m.m., to be diminished by half the thickness of the film, that is by 0,1 m.m.).

By multiplying these values by 0,808, the square of the proportion of the wave-lengths of Cu_{K_α} and Cu_{K_β}, the values which might originate from β-radiation, can be separated, and so a list can be composed which contains α-values exclusively. By using a general method of computation, in which we proceeded from an arbitrary system of axes and on which we shall perhaps publish another paper the grey tin appeared to be regular with great probability. In that case our values must fulfill the equation:

$$sin^2 \frac{\vartheta}{2} = A(h_1^2 + h_2^2 + h_3^2) = \frac{\lambda^2}{4a^2}(h_1^2 + h_2^2 + h_3^2),$$

in which A is a constant, h_1, h_2, and h_3 indices of MILLER, λ wave-length of the Röntgen-radiation, a edge of elementary cube.

So we have only to deduce the constant A and from that the number of atoms in the elementary cube a^3. A provisional calcu-

TABLE I.

Distamces in 0.1 mm.	$\sin^2\frac{\vartheta}{2}\cdot 10^3$	$Cu\ K_\alpha$-radiation				$Cu\ K_\beta$-radiation		
		$\sin^2\frac{\vartheta}{2}\cdot 10^3$ calculated	Intensity	$h_1 h_2 h_3$	$h_1^2+h_2^2+h_3^2$	$\sin^2\frac{\vartheta}{2}\cdot 10^3$ calculated	$h_1 h_2 h_3$	$h_1^2+h_2^2+h_3^2$
f 112	41					34	1 1 1	3
f 144[5]	69	43	1.3	1 1 1	3			
vf 168	92					92	2 2 0	8
m 191	118	114	1.5	2 2 0	8	126	3 1 1	11
m 221	156	156	1.1	3 1 1	11			
vf 239	181					184	4 0 0	16
f 272	230	231	0.4	4 0 0	16			
m 300	274[5]	270	0.6	3 3 1	19	275[5]	4 2 2	24
s 342	346	341	1.0	4 2 2	24			
m 370	395[5]	384	0.6	3 3 1 5 1 1	27	402	5 3 1	35
f 406	461	455	0.4	4 4 0	32	459	6 2 0	40
m 426	498	497	0.7	5 3 1	35			
s 464	567	568	0.6	6 2 0	40			
f 488	611	611	0.3	5 3 3	43			
f 509	648					643	6 4 2	56
f 530	684	682	0.2	4 4 4	48	677	5 5 3 7 3 1	59
m 554	725	725	0.5	5 5 1 7 1 1	51			
vs 596	791	796	0.9	6 4 2	56			
vs 626	834	838	0.6	5 5 3 7 3 1	59			
vf 655	871[5]					862	5 5 5 7 5 1	75
f 678	898	909	0.1	8 0 0	64			
f 694	915					918	8 4 0	80
m 730[5]	949	952	0.2	7 3 3	67			

lation pointed to about eight atoms per cube, which suggests the structure of diamond. So we tried to get agreement of our interference-lines with the structure of diamond.

Using the atomic-weight (119,0) and the specific weight of grey tin at 18° C. (5,751), AVOGADRO's number ($6{,}06.10^{23}$) and the wave-length of the K-series of copper-radiation (α $1{,}541.10^{-8}$, β $1{,}385.10^{-8}$) $A_\alpha = 0{,}0141$ and $A_\beta = 0{,}0114$ are found. Nearly in agreement with this we find as a mean from the observed lines $A_\alpha = 0{,}01422$ and $A_\beta = 0{,}01149$. The third column gives the values of $sin^2 {}^1/_2\theta$, computed with the last-mentioned values of A_α, so far as they give occasion to interference-lines.

From DEBIJE's structure-factor it can be deduced namely, that for the structure of diamond interference-lines can only be got, when h_1, h_2 and h_3 are all even or all odd, and when in case they are even, their sum can be divided by 4. The expression $h_1^2 + h_2^2 + h_3^2$ then becomes 3, 8, 11 and so on, see the sixth column; in the fifth column the corresponding values of h_1, h_2 and h_3 are given.

The intensity, given in the fourth column is found by dividing $64\,(h_1^2 + h_2^2 + h_3^2)$ in the product of the number of planes and the square of the absolute value of the structure-factor. To this intensity only so much value must be attached that only the intensities of three successive lines must be compared.

The seventh, eighth, and ninth columns relate to β-lines.

On comparing the observed and computed values of $sin^2 {}^1/_2\theta$ a satisfactory agreement appears to exist, also in connection with the intensity.

The two smallest squares of sines are not satisfactory [1]). About this Comm. N°. 2*a* p. 498 gives information.

When it is examined whether all lines, which must be expected on the ground of the structure mentioned, are present, it appears, that of the expected α-lines not a single is lacking; some of the β-lines however appearingly have too small an intensity, and so are not observed on the Röntgenogram.

From the values of A_α and A_β, deduced from the observations, we found for a, the edge of the elementary cube $6{,}46.10^{-8}$ cm. at 18° C. For the distance of two nearest atoms $2{,}80.10^{-8}$ cm. is thus deduced.

In the grey tin with its diamond-structure, which silicium possesses too, the tetravalency clearly makes its appearance, whilst in the tetragonal modification, in which each atom is surrounded by

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[1]) Comp. P. DEBIJE, Physik. ZS. 18 (1917), p. 488. Note.

two nearest ones with smaller distances than those of the rest of the surrounding atoms, two valencies especially come to the foreground.

In this connection it might be interesting, to investigate, whether the deviating conduct, observed by GRÜNEISEN [1]) in the course of the conductivity at low temperatures for cadmium, tin, and mercury, and which he ascribed to the metals not being regular, is not found with the grey tin, or whether the diamond-structure still causes a deviation in the conductivity at low temperatures from the conduct of the metals crystallising in cubes with centered faces.

We heartily thank Prof. KEESOM for the great interest, which he always showed us.

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[1]) E. GRÜNEISEN. Verh. D. physik. Ges. 20 (1918), p. 36.

Physics. — *"On the equivalent of parallel translation in non-Euclidean space and on* RIEMANN'S *measure of curvature."* By Dr. A. D. FOKKER. (Communicated by Prof. H. A. LORENTZ.)

(Communicated in the meeting of April 26, 1918).

1. *Introduction.* In the following pages I shall try to give a mental picture of some ideas recently developed by prof. J. A. SCHOUTEN before the Mathematical Society at Amsterdam which will help to illustrate the meaning of a "system of axes moving geodesically", and the "geodesic differential", together with a few applications. [1]) The great point will be to realise in a new way wat kind of displacement in non-Euclidean space must be considered to correspond to a parallel translation, this being an operation indispensable in vector-analysis to compare vectors in different points.

One of the characteristic properties of pure translations is this, that all points of a rigid body are thereby transferred over an equally long distance. This property might be used to define a parallel translation, provided the rigid consists of a number of points exceeding a certain minimum. If, for example, in three-dimensional space, we give a prescribed displacement to one of the points of a rigid system consisting of two or three points, it is not enough to demand an equal displacement for the other point or points to define a translated position without ambiguity. But in a Euclidean space of n dimensions other motions than pure translations are excluded, if for a rigid body of no less than $(2n-2)$ points we want all points to run through equal distances.

This will be our starting-point. We know, however, that in general no body of finite dimensions can move in curved space without changing the mutual distances of its points. In order to retain the idea of a rigid body we shall have to confine ourselves to bodies with dimensions of the order of an infinitesimal ε.

Another and more serious difficulty arises from the fact, that we cannot get all points to shift over exactly the same infinitesimal distance Δ. We cannot but leave a margin of the order of $\Delta\varepsilon^2$ for the separate distances. Here the question arises whether in a certain

[1]) Cf. a treatise offered by Prof. SCHOUTEN to be published in the transactions of the Kon. Akademie: "*Die directe Analysis zur neueren Relativitätstheorie*".

direction only one displacement can be effected in which this approximation to the exact equality is realised? This, however, cannot be expected, since in the special case of Euclidean space not only pure translations but screw-displacements too are allowed by leaving this margin. Therefore a second property of pure translations is required, fit to exclude these screw-displacements.

This property is found in the fact that the shifts are not only equal, but also parallel to one another. This amounts to a certain reciprocity between translations in different directions. Consider two translations, by which a point P is transferred to neighbouring points Q and R respectively. The first translation will carry point R to the same place where the second translation will carry point Q. This property indeed excludes screw-displacements.

In the following pages we shall first give a summary of the results arrived at in this paper, and afterwards (§ 6) give the analytical formulae. For examples we will mainly take those of three-dimensional space. The results, however, will hold good, independent of whatever number (n) of dimensions we choose to ascribe to our space.

2. *Geodesic displacement.* Let us define an infinitesimal *rigid* as *an aggregate of particles, which keep their mutual distances unchanged* during their motions. One of these points we may choose as a central, and imagine the other points defined by the ends of infinitesimal vectors from this central point, these vectors having constant lengths (of the order ε) and including constant angles. The number of points must be no less than $(2n-2)$, hence the number of vectors $(2n-3)$, no n of them being situated together in a space of $(n-1)$ dimensions.

We imagine this rigid to execute motions so as to shift the central particle from a starting point P to neighbouring points over distances of the order Δ.

It appears possible (§ 7) to indicate a certain variety of motions in which, firstly *the shifts of all the other points of the rigid*, up to a margin of the order $\Delta\varepsilon^2$, *equal the shift of the central point*, and, secondly, there exists a certain *reciprocity* which becomes apparent when we observe *two arbitrarily chosen motions belonging to the variety*, which shift the central particle, let us say, from P to Q and from P to R, and when we notice the displacements of the particles having their starting points in R and Q respectively. The *particle from R in the motion (PQ) will reach the same point attained by the particle from Q in the other motion (PR)*.

The two conditions specified determine without ambiguity a variety

of motions which we may call "*geodesic displacements*" of the infinitesimal rigid. They are the substitutes for parallel displacements[1]) in Euclidean extensions. We may assign the name "compass-rigid" to a small rigid body that cannot move but in the geodesic manner defined. It must be understood that a compass-rigid which, after a displacement, returns to its starting-point by the same way, will on arrival be in its starting-position too. If, however, it returns by a circuit, it generally will not be in its starting-position again on arrival.

3. *Geodesic differential.* If we want to compare two vectors in neighbouring points P and Q, we can proceed as follows. We put a compass-rigid with its centre in P and by marking one of its points we delineate the vector in it. Now displacing the compass-rigid to Q it is reasonable to say that the marked point defines the vector displaced geodesically from P to Q. By comparing this vector with the one present in Q we immediately see the meaning of the *geodesic differential* of a vector. If this is known, it is clear what CHRISTOFFEL's covariant differentiation means.

In the same way we can displace our vector-units from P to Q. In general these will differ from the vector-units in Q. A set of geodesically displaced vector-units is what Prof. SCHOUTEN defined as a system of axes moving geodesically.

4. *Geodesic line.* We can easily imagine what we have to do in order to *prolong a given line-element geodesically.* We put the centre of the compass-rigid in the starting-point and mark the end of the line-element by an arrow in the compass-rigid. After the centre has been displaced along the line-element, the arrow will point in another definite direction. This is the geodesic prolongation of the element. Continuing to move the compass-rigid in the direction of the arrow, the centre will gradually describe a geodesic line.

In this case, during displacements along a geodesic line, vectors moving geodesically will continue to include constant angles with the geodesic (cf. LEVI-CIVITA's article), these angles being fixed angles in the compass-rigid.

5. RIEMANN's *measure of curvature.* Let us now suppose that we

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[1]) Taking another starting-point, T. LEVI-CIVITA arrives at a definition of parallelism which comes to the same thing: "*Nozione di parallelismo in una varieta qualunque, e conseguente specificazione geometrica della curvatura Riemanniana*". Rend. Circ. Mat. Palermo, XLII. p. 1, 1917. His geometrical explanation of the measure of curvature, however, is totally different from the one we shall give in section 5.

make a compass-rigid describe a small circuit, e.g. along a vanishing (quasi-)parallelogram. We already pointed out that in general it will *not* return to its starting-*position*. The difference between the two positions is such as might have been produced by an infinitesimal rotation around the starting-point. The amount of this rotation is proportional to the area of the circuit described, the orientation of the "axis" of rotation (which in higher extensions is of $(n-2)$ dimensions) being determined by the orientation of the plane of the circuit. The rotation is intimately connected with the curvature of space. When this *rotation of curvature*, as it may be called, vanishes in all points for every arbitrary circuit, then the space is Euclidean. [1])

The components of the operator by which from the data of the area included by the circuit the rotation of curvature for the compass-rigid can be derived, are RIEMANN'S four-index-symbols, of the second kind.

Further — to confine ourselves to three-dimensional space — if we take the length of the axis of rotation equal to the amount of the angle of rotation, and construct a parallelepiped with this axis and the parallelogram of the circuit, we can consider the ratio of its volume to the square of the parellelogram as a measure for the curvature of space. Indeed, in the limit, for a vanishing circuit, this ratio is just the number indicated by RIEMANN as *the measure of curvature of the space with respect to the plane of the circuit considered.*

6. Now we shall proceed to give the required formulae. We take the length of a line-element as defined by

$$ds^2 = \Sigma(ab)\, g_{ab} dx^a\, dx^b ,$$

$dx^a\ dx^b$ representing increments of the coordinates along the line-element **dx**, $g_{ab}\ (= g_{ba})$ being regular functions of the coordinates of the starting-point, each index in the sum going through all the values from 1 to n, where n is the number of dimensions of space. The algebraical complements of g_{ab} will be denoted by g^{ab}, so that

[1]) The fundamental idea of a recent article by H. WEYL (*Gravitation und Elektrizität*, Berl. Sitz. Ber. May, 1918) may be considered the hypothesis that a small rigid (= "Vektorraum") after turning about an infinitesimal circuit of "translations" (of a somewhat more general kind) not only will have got in a changed position, but in general will have *changed its dimensions* as well. In four-dimensional space-time the linear dilatation of the (4-dimensional) rigid would be half the scalar product of the alternating electromagnetic tensor and the area included by the circuit. (Note added during the revisal of the proofs).

$$\Sigma(b) g_{ab} g^{nb} = \begin{cases} 1 \text{ for } n = a, \\ 0 \text{ for } n \neq a \end{cases}$$

For the sake of brevity we shall write g for the determinant formed of the g^{ab}. Further we shall avail ourselves of CHRISTOFFEL's well-known symbols:

$$\begin{bmatrix} l\,m \\ a \end{bmatrix} = \tfrac{1}{2}\left[\frac{\partial g_{am}}{\partial x^l} + \frac{\partial g_{la}}{\partial x^m} - \frac{\partial g_{lm}}{\partial x^a}\right], \qquad \begin{Bmatrix} l\,m \\ b \end{Bmatrix} = \Sigma(a)\, g^{ab} \begin{bmatrix} l\,m \\ a \end{bmatrix}.$$

The definition of the line-element entails the definition of the length of a vector **v**, with components v^a:

$$v^2 = \Sigma(ab)\, g_{ab}\, v^a\, v^b,$$

and the definition of the angle between two vectors **v** and **w**:

$$vw \cos(vw) = \Sigma(ab)\, g_{ab}\, v^a\, w^b.$$

7. Let the points of a small rigid be given by their coordinates relative to the centre: u^a, v^a, w^a ... ($a = 1, 2, 3, n$), these being the components of vectors **u**, **v**, **w** ... which are of the order of a vanishing quantity ε. If the centre shifts from P to a neighbouring point Q, determined by the infinitesimal increments in the coordinates dx^a (of the order Δ), then we require the new coordinates of the points of the rigid relative to Q in order to satisfy the definition and first condition of section 2: the points are to be points of a rigid, and must each cover an equal distance.

Denoting the new relative coordinates by $u^a + du^a$, $v^a + dv^a$ ... etc. it is easy to formulate the latter half of the condition. For the increments of the coordinates of the point designed by **u** will be $dx^a + du^a$, and the starting point of the line-element through which it runs lies beside P, at a distance defined by **u**. So, if this line-element is to be equal to that from P to Q, i.e.

$$ds^2 = \Sigma(ab)\, g_{ab}\, dx^a\, dx^b,$$

we necessarily must have

$$0 = \Sigma(ab\,m) \frac{\partial g_{ab}}{\partial x^m} u^m\, dx^a\, dx^b + \Sigma(ab)\, 2 g_{ab}\, dx^a\, du^b \quad . \quad . \quad (1)$$

If the aggregrate of points is to form a rigid, both the lengths of the relative vectors **u**, **v**, **w**.... and the included angles must be constant, and expressions such as

$$u^2 = \Sigma(ab)\, g_{am}\, u^a\, u^m \quad , \quad uv \cos(uv) = \Sigma(am)\, g_{am}\, u^a\, v^m,$$

must have the same value in P and Q. This implies

$$0 = \Sigma(a\,m\,l) \frac{\partial g_{am}}{\partial x^l} dx^l\, u^a\, u^m + \Sigma(am)\, 2 g_{am}\, u^a\, du^m, \quad . \quad . \quad . \quad (2)$$

and

$$0 = \Sigma\,(a\,m\,l)\,\frac{\partial g_{am}}{\partial x^l}\,dx^l u^a\,v^m + \Sigma(am)\,g_{am}\,\{u^a\,dv^m + v^m\,du^a\} \quad . \quad (3)$$

These are the equations which must be applicable to du^a, dv^a. . . etc. in the translations mentioned. It is not difficult to find espressions satisfying the equations. We can add to (1) the identity

$$0 \equiv \Sigma\,(a\,l\,m)\left(\frac{\partial g_{am}}{\partial x^l} - \frac{\partial g_{lm}}{\partial x^a}\right) dx^a\,dx^l\,u^m;$$

and a similar identity to (2):

$$0 \equiv \Sigma\,(a\,l\,m)\left(\frac{\partial g_{al}}{\partial x^m} - \frac{\partial g_{lm}}{\partial x^a}\right) u^a\,dx^l\,u^m.$$

Replace at the same time the index b in the first term, right-hand side of (1) by l, and m in the second term, right-hand side of (2), by b, and we get

$$0 = \Sigma\,(a)\left[\Sigma\,(lm)\left(\frac{\partial g_{am}}{\partial x^l} + \frac{\partial g_{al}}{\partial x^m} - \frac{\partial g_{lm}}{\partial x^a}\right) dx^l\,u^m + \Sigma(b)\,2g_{ab}\,du^b\right] dx^a,$$

and

$$0 = \Sigma\,(a)\left[\Sigma(lm)\left(\frac{\partial g_{am}}{\partial x^l} + \frac{\partial g_{al}}{\partial x^m} - \frac{\partial g_{lm}}{\partial x^a}\right) dx^l\,u^m + \Sigma(b)\,2g_{ab}\,du^b\right] u^a.$$

Dividing by 2 we can reduce the equations to the form

$$0 = \Sigma\,(ab)\,g_{ab}\,dx^a\left[du^b + \Sigma(lm)\begin{Bmatrix} l\,m \\ b \end{Bmatrix} dx^l\,u^m\right], \quad . \quad . \quad . \quad (1')$$

$$0 = \Sigma\,(ab)\,g_{ab}\,u^a\left[du^b + \Sigma(lm)\begin{Bmatrix} l\,m \\ b \end{Bmatrix} dx^l\,u^m\right], \quad . \quad . \quad . \quad (2')$$

Similarly, we can put for the third equation

$$0 = \Sigma(ab)\,g_{ab}\left[u^a\,[dv^b + \Sigma\begin{Bmatrix} lm \\ b \end{Bmatrix} dx^l\,v^m] + v^a\,[du^b + \Sigma\begin{Bmatrix} lm \\ b \end{Bmatrix} dx^l\,u^m]\right] \quad (3')$$

So we can satisfy the imposed condition by taking

$$du^b = -\,\Sigma(lm)\begin{Bmatrix} l\,m \\ b \end{Bmatrix} dx^l\,u^m. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and similar expressions for dv^b, dw^b. . . .

The equation (4) is covariant. It will retain its form wathever be the choice of coordinates.

It defines the position of the points of the small rigid when, by a first approximation, they have all covered the same infinitesimal distance.

It is seen, from (1), that in developing g_{ab} into a series we have neglected terms with products $u^m\,u^n$. The squares of the distances covered therefore can differ from $P\,Q$ by an amount of the order $\varepsilon^2\,\Delta^2$, so that the distances may only be taken as equal up to an amount of the order $\varepsilon^2\,\Delta$, which we shall neglect.

8. The "corrections" given by eq. (4) are of the order $\Delta\varepsilon$. In order to see if the solution defined by them is the only one, we may ask if we can satisfy the equations by "corrections" differing from du^a, such as $du^a + \delta u^a$, where δu^a is of the same order as du_a.

If these are to satisfy eq. (1), (2), (3), we must evidently have.

$$0 = \Sigma(ab)\, g_{ab}\, dx^a\, \delta u^b ,$$
$$0 = \Sigma(ab)\, g_{ab}\, u^a\, \delta u^b ,$$
$$0 = \Sigma(ab)\, g_{ab}\, u^a\, \delta v^b + g_{ab}\, v^a\, \delta u^b , \text{ etc.}$$

If the rigid, besides the centre, consists of p points, we shall have $2p + \frac{1}{2}p(p-1)$ equations for pn variables δu^a, $\delta v^a \ldots$ etc. For $p = 2n-3$ we have as many homogeneous linear equations as there are variables. If no set of n vectors **u,v,w**. . . are situated together in an $(n-1)$-dimensional space, then these equations only permit a solution of the form (e.g. for $n = 3$):

$$\delta u^a = \frac{\omega}{\sqrt{g}} \Sigma(ij) \begin{vmatrix} g_{bi} & g_{ci} \\ g_{bj} & g_{cj} \end{vmatrix} dx^i\, u^j = \frac{\omega}{\sqrt{g}} \begin{vmatrix} dx_b & dx_c \\ u_b & u_c \end{vmatrix} , \quad . \quad . \quad (5)$$

where ω is an arbitrary number, and by b, c, a, we mean a set of three indices which form an even permutation of 1, 2, 3. We denote by dx_b and u_b with lowered index the covariant combinations:

$$dx_b = \Sigma(i)\, g_{bi}\, dx^i \quad , \quad u_b = \Sigma(j)\, g_{bj}\, u_j .$$

It can easily be ascertained that the expression given for δu^a, together with similar expressions for δv^a, δw^a. . . satisfy the equations. They must define the displacements in the case of an infinitesimal rotation about **dx** as an axis [1]). For all vectors **u,v,w** . . . keep their lengths unchanged and both the angles included with **dx** and the mutual angles remain unaltered.

Since the condition imposed thus far appears not to be sufficient to define a displacement without ambiguity, we must recur to the condition of reciprocity of section 2.

Shifting the centre of the compass-rigid from P to Q the particle designed by **u** might come from a position R into the position defined by the coordinates

$$x^a_P + dx^a + u^a + du^a + \delta u^a ,$$

or,

$$x^a_P + dx^a + u^a - \Sigma(lm) \left\{ \begin{matrix} lm \\ a \end{matrix} \right\} dx^l u^m + \frac{\omega}{\sqrt{g}} \begin{vmatrix} dx_b & dx_c \\ u_b & u_c \end{vmatrix} .$$

If we now shift the centre from P to R, and we then wish to find what will be the new position for the particle from Q according

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[1]) Through an angle $\omega|dx|$.

to the same displacement-law, we only have to interchange the vectors **dx** and **u**.

Now we see, the determinant changing its sign, that this position will never be the same as that reached by the former particle from R, unless $\omega = 0$.

So the application of the condition of reciprocity excludes screw-movements [1]).

9. Now in the following way we can see that the condition of the body's rigidity and the equality of the covered distances together with the condition of reciprocity are sufficient to define the variety of geodesic displacements without ambiguity.

From eq. (1) and (2) we learn that the required "corrections" du^a must be proportional both to the components of the displacement **dx** and of the vector **u**. Therefore let us put

$$du^a = \Sigma\, h^a_{st}\, dx^s\, u^t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4')$$

Now, according to the condition of reciprocity we must apparently have

$$h^a_{st} = h^a_{ts}\,.$$

Substitute (4') in (3), and we get

$$0 = \Sigma\,(a\,l\,m)\,\frac{\partial g_{am}}{\partial x^l}\,dx^l\,u^a\,v^m + \Sigma\,(a\,m\,s\,t)\,g_{am}\,\{h^m_{st}\,u^a\,dx^s\,v^t + h^a_{st}\,v^m\,dx^s\,u_t\}.$$

Taking other indices and putting

$$h_{a,lm} = \Sigma(b)\,g_{ab}\,h^b_{lm}\,, \qquad (h_{a,lm} = h_{a,ml}),$$

we get

$$0 = \Sigma\,(a\,l\,m)\,dx^l\,u^a\,v^m\left\{\frac{\partial g_{am}}{\partial x^l} + h_{a,lm} + h_{m,la}\right\}.$$

In this equation we may regard the forms in brackets as unknown variables. Because of the symmetry in the indices a and m there are $\frac{1}{2}\,n^2\,(n-1)$ of them. As the equation is to hold for an arbitrary combination

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[1]) Dr. DROSTE remarked to me that a screw-motion might be excluded in a different manner. Let PQ be part of a geodesic. In P and in Q take two infinitesimal planes perpendicular to the geodesic. Draw the geodesics perpendicular to the first plane and intersecting it in a line-element PR. These together form a "geodesic strip", which will intersect the second plane in an element QR'. PR and QR' can be called "parallel" and in the same way each pair of elements in the same geodesic strip including equal angles with the geodesic PQ.

In our chain of thought, however, geodesic lines are defined by making use of the idea of geodesic displacements (see section 10), and so we cannot avail ourselves of the above suggestion.

of three vectors **dx, u, v,** we conclude that the variables must vanish. Thus

$$0 = \frac{\partial g_{am}}{\partial x^l} + h_{a,lm} + h_{m,la}.$$

Similarly

$$0 = \frac{\partial g_{la}}{\partial x^m} + h_{l,ma} + h_{a,ml}$$

$$0 = \frac{\partial g_{lm}}{\partial x^a} + h_{m,al} + h_{l,am}.$$

By adding the first two, and subtracting the last, and considering that $h_{l,am} = h_{l,ma}$ etc. we find

$$h_{a,lm} = -\begin{bmatrix} lm \\ a \end{bmatrix}.$$

Now we know

$$h^b_{lm} = \Sigma\, g^{ab}\, h_{a,lm} = -\Sigma\, g^{ab} \begin{bmatrix} lm \\ a \end{bmatrix} = -\begin{Bmatrix} lm \\ b \end{Bmatrix},$$

and so from (4') we see that our values for du^b:

$$du^b = -\Sigma(lm) \begin{Bmatrix} lm \\ b \end{Bmatrix} dx^l u^m \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

constitute the only solution consistent with all conditions.

10. To explain the applications of sections 3 and 4 we proceed as follows. Suppose a vector **V,** in point P, be marked in the compass-rigid. After a geodesical displacement to Q the marked vector will have got the components

$$V^a - \Sigma \begin{Bmatrix} lm \\ a \end{Bmatrix} dx^l V^m.$$

Now if we have in Q a vector with components $V^a + dV^a$, where dV^a now represents some increment of the component V^a, then obviously the components of the *geodesic differential* are

$$dV^a + \Sigma(lm) \begin{Bmatrix} lm \\ a \end{Bmatrix} dx^l V^m.$$

This geodesic differential will be a vector itself, being the difference of two vectors, while dV^a are no vector-components.

If the line-element PQ itself is drawn in the compass-rigid as a vector with components dx^a, and displaced geodesically, then in Q the arrow will have got components

$$dx^a - \Sigma(lm) \begin{Bmatrix} lm \\ a \end{Bmatrix} dx^l dx^m.$$

This arrow we have called the *geodesic prolongation* of the

element. It is easily seen that this entails for the geodesic the equation

$$0 = d^2x^a + \Sigma(lm)\left\{\begin{matrix} lm \\ a \end{matrix}\right\} dx^l\, dx^m .$$

This (covariant) equation coïncides with what we get from the familiar definition of a geodesic as the shortest line between two points.

11. We shall now displace geodesically a particle P', which, with relation to P, is defined by a vector $\mathbf{u}$, to a point S' near T, by shifting the compass rigid in two steps from P to T, along PQ and QT. Then, a second time, we displace the particle to S'' near T, taking the steps along PK and KT, the quadrilateral $PQTK$ being a (quasi-) parallelogram with sides $\mathbf{dx}$ (PQ and KT) and $\delta\mathbf{x}$ (PK and QT).

After the displacement along PQ the coordinates of the particle considered relative to Q have become

$$u^a - \Sigma(lm)\left\{\begin{matrix} lm \\ a \end{matrix}\right\} dx^l\, u^m .$$

At the second step we must be careful to take the values of CRISTOFFEL's symbols at point Q, so that after the displacements along PQ and QT the coordinates relative to T are

$$u^a - \Sigma\left\{\begin{matrix} lm \\ a \end{matrix}\right\} dx^l u^m - \Sigma\left\{\begin{matrix} lm \\ a \end{matrix}\right\} \delta x^l \left[u^m - \Sigma\left\{\begin{matrix} pq \\ m \end{matrix}\right\} dx^p u^q \right] - \Sigma \frac{\partial}{\partial x^p}\left\{\begin{matrix} lm \\ a \end{matrix}\right\} dx^p \delta x^l u^m .$$

If the displacements had been performed along PK and KT, the coordinates relative to T would have been

$$u^a - \Sigma\left\{\begin{matrix} lm \\ a \end{matrix}\right\} \delta x^l u^m - \Sigma\left\{\begin{matrix} lm \\ a \end{matrix}\right\} dx^l \left[u^m - \Sigma\left\{\begin{matrix} pq \\ m \end{matrix}\right\} \delta x^p u^q \right] - \Sigma \frac{\partial}{\partial x^p}\left\{\begin{matrix} lm \\ a \end{matrix}\right\} \delta x^p dx^l u^m .$$

Taking the difference we find

$$\zeta^a = \Sigma(lmp)\left[\frac{\partial}{\partial x^p}\left\{\begin{matrix} lm \\ a \end{matrix}\right\} - \Sigma(n)\left\{\begin{matrix} l\,n \\ a \end{matrix}\right\}\left\{\begin{matrix} pm \\ n \end{matrix}\right\} \right] (dx^l\, \delta x^p - dx^p\, \delta x^l)\, u^m .$$

or

$$\zeta^a = \tfrac{1}{2}\Sigma(lmp)\left[\frac{\partial}{\partial x^p}\left\{\begin{matrix} lm \\ a \end{matrix}\right\} - \frac{\partial}{\partial x^l}\left\{\begin{matrix} pm \\ a \end{matrix}\right\} + \Sigma\left\{\begin{matrix} pn \\ a \end{matrix}\right\}\left\{\begin{matrix} lm \\ n \end{matrix}\right\} - \Sigma\left\{\begin{matrix} ln \\ a \end{matrix}\right\}\left\{\begin{matrix} pm \\ n \end{matrix}\right\} \right] \times$$
$$\times (dx^l\, \delta x^p - dx^p\, \delta x^l)\, u^m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The first factor is seen to be a RIEMANN's four-index-symbol, of the second kind. Availing ourselves of his notation we can put

$$\zeta^a = \tfrac{1}{2}\Sigma(lmp)\left\{ma, lp\right\}(dx^l\, \delta x^p - dx^p \delta x^l)u^m \quad . \quad . \quad . \quad . \quad (6)$$

The ζ^a ($a = 1, 2, \ldots n$) are the components of the displacement-

vectors which would become manifest after a geodesical displacement of the compass-rigid about the circuit $TKPQT$. The displacement cannot be anything else but a rotation, the lengths and angles remaining the same. We see how RIEMANN's symbols determine the rotation in terms of the components of the area of the circuit. This rotation is characteristic of the curvature of space.

12. It now remains to prove the statement of section 5 as to the interpretation of RIEMANN's measure of curvature.

The measure of curvature with respect to the plane of **dx** and **δx** is defined to be [1])

$$K = \frac{\Sigma(a\,lp\,mq)\,g_{aq}\,\{ma.\,lp\}\,(dx^l\,\delta x^p - dx^p\delta x^l)(dx^m\,\delta x^q - dx^q\delta x^m)}{\Sigma(lp\,mq)(g_{lm}g_{pq} - g_{lq}\,g_{pm})(dx^l\,\delta x^p - dx^p\delta x^l)(dx^m\,\delta x^q - dx^q\delta x^m)}.$$

The denominator is four times the sqnare of the area of the parallelogram formed by **dx** and **δx**. For by changing indices without changing the sum we get four times

$$\Sigma(lp\,mq)\begin{vmatrix} g_{lm} & g_{lq} \\ g_{pm} & g_{pq} \end{vmatrix} dx^l\,dx^m\,\delta x^p\delta x^q.$$

Writing d for the length of **dx**, and δ for the length of **δx**, we find for the denominator

$$4\begin{vmatrix} d^2 & d\delta\cos(d\delta) \\ d\delta\cos(d\delta) & \delta^2 \end{vmatrix},$$

this being four times the square of the area of the parallelogram.

We shall discuss the numerator for the case of three-dimensional space and show that it represents four times the volume of the parallelepiped formed by the axis of rotation and the parallelogram.

Proceeding with some caution, the analogon in more-dimensional cases is readily found in the same way. We will put for the numerator

$$2\,\Sigma(a\,m\,q)\,g_{aq}\,R^a_m\,(dx^m\,\delta x^q - dx^q\delta x^m) \quad . \quad . \quad . \quad . \quad . \quad (7)$$

denoting by $R^a{}_m$ the coefficients of the rotation of curvature (6):

$$\zeta^a = \Sigma(m)\,R^a_m\,u^m.$$

or,

$$\zeta^c = \Sigma(j)\,R^c_j\,u^j.$$

How are the numbers R^c_j related to the components of the axis of rotation? If we suppose the components of the latter equal to l^i, then the rotation is represented, as will be presently shown, by

$$\zeta^c = \frac{1}{\sqrt{g}}\,\Sigma(ij)\begin{vmatrix} g_{ai} & g_{bi} \\ g_{aj} & g_{bj} \end{vmatrix} l^i\,u^j \quad . \quad . \quad . \quad . \quad . \quad (8)$$

[1]) Cf. for example BIANCHI, Lect. on Diff. Geometry, section 319.

Here we mean by c that index, which with a and b forms an even permutation of 1 2 3. By pqr we shall denote a similar set:

$$abc\ (=)\ pqr\ (=)\ 123.$$

We already saw that displacements of this kind constitute a rotation. To inquire whether the angular amount of the rotation is equal to the length of $\mathbf{l}$, we must observe the displacement of the end of a vector $\mathbf{u}$ which is perpendicular to $\mathbf{l}$, so that

$$\Sigma\,(ab)\,g_{ab}\,u^a\,l^b = 0\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This displacement ought to be $|\mathbf{u}|$ multiplied by $|\mathbf{l}|$. Let us calculate $|\zeta|^2$:

$$\zeta^2 = \Sigma(cr)\,g_{cr}\zeta^c\,\zeta^r,$$

$$= \frac{1}{g}\,(rij\,v\,w)\begin{vmatrix} g_{ai} & g_{bi} & g_{ci} \\ g_{aj} & g_{bj} & g_{cj} \\ g_{ar} & g_{br} & g_{cr} \end{vmatrix} l^i\,u^j\,. \begin{vmatrix} g_{pv} & g_{qv} \\ g_{pw} & g_{qw} \end{vmatrix} l^v u^w.$$

The summation with respect to c has been effected by writing in full the first determinant. If we want to sum up with respect to r, we notice that the determinant vanishes but for one special value of r, which is different both from i and j. If $i = p$, $j = q$, then the determinant becomes $+g$, if $i = q$, $j = p$, then we get $-g$. In both cases we may write

$$\zeta^2 = \Sigma\,(pq\,vw)\begin{vmatrix} g_{pv} & g_{qv} \\ g_{pw} & g_{qw} \end{vmatrix} l^p\,l^v\,u^q\,u^w,$$

and, by (9):

$$\zeta^2 = u^2\,l^2,$$

So the correctness of formula (8) has been shown.

But then we are justified in putting

$$R^c_j = \frac{1}{\sqrt{g}}\,\Sigma(i)\begin{vmatrix} g_{ai} & g_{bi} \\ g_{aj} & g_{bj} \end{vmatrix} l^i,$$

and we can subsequently show that (7) represents four times the parallelepiped mentioned. We write (7) with a slight alteration of indices and we get:

$$2\,\Sigma\,(c\,j\,k)\,g_{ck}\,R^c_j\ (dx^j\,\delta x^k - dx^k\delta x^j) =$$

$$= \frac{2}{\sqrt{g}}\ \Sigma(ijk)\begin{vmatrix} g_{ai} & g_{bi} & g_{ci} \\ g_{aj} & g_{bj} & g_{cj} \\ g_{ak} & g_{bk} & g_{ck} \end{vmatrix} l^i\,(dx^j\,\delta x^k - dx^k\delta x^j).$$

Now if j and k assume all values, a set j, k furnishes just as much as a set k, j, the determinant taking the value $+g$ or $-g$ according to the combination i, j, k being an even or odd permutation

of 1 2 3, and vanishing for other combinations. So we get for the numerator

$$4\sqrt{g}\begin{vmatrix} l^i & l^j & l^k \\ dx^i & dx^j & dx^k \\ \delta x^i & \delta x^j & \delta x^k \end{vmatrix}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

this being four times the volume of the parallelepiped formed by **l**, **dx** and δ**x**.

This sufficiently explains section 5. We may remark that formula (8) for the displacements of rotation implies a convention as to the direction in which the axis of rotation has to be drawn. The axis of rotation must be orientated in a manner to ensure that the direction of ζ is correlated to the directions of **l** and **u**, i.e. a parallelepiped constructed from **l**, **u** and ζ, in the order thus specified, must have a positive volume:

$$\sqrt{g}\begin{vmatrix} l^a & l^b & l^c \\ u^a & u^b & u^c \\ \zeta^a & \zeta^b & \zeta^c \end{vmatrix} = \textit{positive}.$$

This amounts to the same relation which exists between the directions of the positive axes of X, Y, Z.

One sees from (10) that the measure of curvature will be positive if the direction of the axis of the rotation of curvature bears the above-mentioned correlation to the directions of **dx** and δ**x**.

Similarly, in four dimensions, if the axis of a rotation in a special case be a parallelogram on the vectors **l** and **m**, then the rotation is given by

$$\zeta^d = \frac{1}{\sqrt{g}} \Sigma (i\,j\,k) \begin{vmatrix} g_{ai} & g_{bi} & g_{ci} \\ g_{aj} & g_{bj} & g_{cj} \\ g_{ak} & g_{bk} & g_{ck} \end{vmatrix} l^i\, m^j\, u^k .$$

where $a\,b\,c\,d$ (=) 1 2 3 4, and the direction of ζ is correlated to the directions of **l**, **m** and **u**, i.e.

$$\sqrt{g}\begin{vmatrix} l^a & l^b & l^c & l^d \\ m^a & m^b & m^c & m^d \\ u^a & u^b & u^c & u^d \\ \zeta^a & \zeta^b & \zeta^c & \zeta^d \end{vmatrix} = \textit{positive}$$

My thanks are due to prof. J. A. Schouten for his kindness in allowing me to read his treatise on Direct Analysis, which is to be published soon in the Transactions of the Kon. Akademie.

Astronomy. — *"The Distribution of the absolute Magnitudes among the Stars in and about the Milky Way."* (First Communication). By Dr. W. J. A. SCHOUTEN. (Communicated by Prof. J. C. KAPTEYN).

(Communicated in the meeting of October 26, 1918).

1. *Introduction.*

One of the most important problems of statistical astronomy is to examine, how in each part of space the stars with bright and faint absolute magnitudes are mixed and of which percentage of stars the luminosity lies between definite limits. Luminosity means the apparent brightness, which a star would have at the unit of distance, and for this unit we will take, in imitation of Professor KAPTEYN, the distance corresponding to an annual parallax $\pi = 0''.1$.

The first determination of the luminosity law was performed by Professor KAPTEYN and published in *Publ. Groningen* N°. 11 [1]). Afterwards several astronomers, employing different methods, have repeated this investigation. Besides the studies of COMSTOCK and WALKEY, who availed themselves of measured parallaxes, and whose results therefore are not much to be relied upon, unless great cautiousness is used, the researches of SEELIGER and SCHWARZSCHILD are well-known. In our dissertation [2]) we have discussed the three methods mentioned, and we have compared the results that were found. It appeared that serious objections may be raised against SEELIGER's investigation, so that we cannot attach much value to the frequency-function of absolute magnitudes found by him. We think we have also demonstrated in our work just cited that the method of KAPTEYN is for various reasons greatly to be preferred to that of SCHWARZSCHILD.

After we had finished this inquiry an earnest wish arose in us to establish the luminosity law according to the method which we think the best. It was known to us, that such a determination had been

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[1]) Also in these Proceedings, Vol. III, p. 658.

[2]) On the Determination of the Principal Laws of Statistical Astronomy. Amsterdam, Kirchner 1918.

in course of preparation at the Astronomical Laboratory at Groningen for a long time already.

In this preparation are included the countings of stars of determined magnitude which are published in *Publ. Groningen* N°s. 18 and 27, and the mean parallaxes of stars of determined magnitude and galactic latitude, which will soon be published and have been kindly lent us for our use by Prof. KAPTEYN and Dr. VAN RHIJN.

It is a matter of course that the preliminary results obtained by us, should be replaced by those of the definitive solution as soon as these are available. Yet we thought we might publish our results as they are partly based on other data, because they give a notion of the exactness that is to be obtained now already, and our preliminary results might be of some service perhaps until the time that the definitive shall have appeared.

In this communication we mention the results that we found, when we tried to determine the luminosity law according to KAPTEYN's method for the whole sky and for five zones of different galactic latitude.

We intend to publish in a second communication the results that are found, when SCHWARZSCHILD's method is applied to the same data. At the same time we hope to compare our results with those of other investigators.

In this article we also gratefully tender our sincere thanks to Prof. KAPTEYN for his kind help, which favoured our investigation.

2. *The Data of Observation.*

In applying KAPTEYN's method we have to take from the observations the following data:

1. the numbers $N_{m,\mu}$ i.e. the numbers of stars of determined apparent magnitude and proper motion,

2. the mean parallaxes $\pi_{m,\mu}$ of stars of determined magnitude and proper motion,

3. the value of ϱ, the probable error of the error curve $\log. {}^{\pi}/_{\pi_0}$ in which π is the real and π_0 the probable parallax.

In our investigation we divided the sky into 5 zones. The galactic latitude we shall indicate by b.

Zone I	=	part of the sky between	$b = -10°$	and	$b = +10°$.					
Zone II	=	„	„	„	„	with b from	$-10°$	to $-30°$	and	$+10°$ to $+30°$.
Zone III	=	„	„	„	„	„ b „	$-30°$	„ $-50°$	„	$+30°$ „ $+50°$.
Zone IV	=	„	„	„	„	„ b „	$-50°$	„ $-70°$	„	$+50°$ „ $+70°$.
Zone V	=	„	„	„	„	„ b „	$-70°$	„ $-90°$	„	$+70°$ „ $+90°$.

We have derived the numbers $N_{m,\mu}$ from the numbers of stars of determined magnitude and galactic latitude by performing countings in catalogues of stars with known proper motions. We have computed these numbers N_m for the different zones from the tables in *Publ. Groningen* N°. 18, as these have been corrected in *Publ. Groningen* N°. 27.

The results that VAN RHIJN published in the work last-mentioned, are very reliable, as they were confirmed by several independent studies. The results of NORT and SEARES, indeed, agree fairly well with those of VAN RHIJN and the numbers of CHAPMAN and MELOTTE too correspond to the Groningen countings as soon as they shall have been corrected for a mistake in the method of reduction which has been pointed out by VAN RHIJN.

In our countings the stars were divided according to their magnitudes into groups of 1—2.9, 3.0—3.9,4.0—4.9, ... 12.0—12.9. According to their proper motions they were counted between the limits 0—2".9, 3".0—4".9, 5".0—7".9, 8".0—9".9, 10".0—14".9, 15".0—19".9, 20".0—29".9, 30".0—49".9 and >49".9.

The following catalogues have been used:

1. L. Boss. Preliminary Catalogue of 6188 Stars, 1910.

2. A. AUWERS. Catalog der Astronomischen Gesellschaft, Zone + 15° bis + 20°, 1896.

3. W. G. THACKERAY. Greenwich 1910 Catalogue of Stars, Zone + 24°0′ to + 32°0′. Analysis of Number and Percentages of Proper Motions. *Monthly Notices* **77**, 204—212, 1917.

F. W. DYSON and W. G. THACKERAY. The Systematic Motions of the Stars between Dec. + 24° and Dec. + 32°. *Monthly Notices* **77**, 581—596, 1917.

4. F. W. DYSON and W. G. THACKERAY. New Reduction of GROOMBRIDGE's Catalogue of Circumpolar Stars, 1905.

5. J. C. KAPTEYN and W. DE SITTER. The Proper Motions of 3300 Stars of different Galactic Latitudes, derived from photographic plates, prepared by Prof. ANDERS DONNER. *Publ. Groningen* N°. 19, 1908.

6. J. C. KAPTEYN. The Proper Motions of 3714 Stars derived from plates taken at the observatories of Helsingfors and the Cape of Good Hope. With the co-operation of Dr. H. A. WEERSMA. *Publ. Groningen* N°. 25, 1914.

7. Dr. A. A. RAMBAUT. A photographic Determination of the Proper Motion of 250 Stars in the Neighbourhood of Σ 443. *Monthly Notices* **73**, 616—630, 1913.

8. A. VAN MAANEN. Remarks on the Motion of the Stars in and near the double Cluster in Perseus. *Report of the Nineteenth Meeting*

of the American Astronomical Society. Popular Astronomy **25**, 108—110, 1917.

9. W. G. THACKERAY. Notes on some Proper Motions derived from a Comparison of CARRINGTON's Catalogue with the Greenwich Places for 1900. *Monthly Notices* **67**, 146—148, 1906.

Boss' catalogue was only used for stars brighter than $5^m.8$.

AUWERS' catalogue was only used to determine $N_{m,\mu}$ for the whole sky.

The proper motions that have been derived from the Cape plates in *Publ. Groningen* N°. 25, are not very accurate. The plates had not been originally destined for deriving proper motions from them, and they had been measured absolutely. We have, therefore, in determining the mean values attached but very little weight to them.

No corrections have been applied in order to reduce our results for the different catalogues to the same scale of magnitudes or to one fundamental system or in order to correct the proper motions for errors of observation. These corrections may in our opinion be neglected, considering the comparative inaccuracy of the numbers. Moreover, we have always expressed the numbers $N_{m,\mu}$ for every magnitude in percentages of the numbers N_m and these percentages appeared to vary only little with the magnitudes.

Our countings in determining $N_{m,\mu}$ for the whole sky include 38818 proper motions, while moreover in the five zones resp. 8273, 10857, 6981, 3144 and 1488 stars were counted.

Now we are able to determine the numbers of stars of determined magnitude and proper motion. The results we found for the whole sky, are mentioned in table 1.

In our further research we did not use the numbers of stars with $\mu > 50''$. It is very difficult to determine these with sufficient accuracy from the data of known proper motions, so scanty as yet. It will appear that in consequence of this limitation we could not extend the luminosity curve found by us to the faintest stars.

The numbers $N_{m,\mu}$ now being known for the different zones, we may also examine how the stars with determined proper motions are distributed over the sky with reference to the Milky Way. In order to do so we have calculated the numbers of stars with P.M. resp. $> 10''$, $5''$, $3''$ and $0''$ for every magnitude per 100 square degrees. It appeared that the numbers of stars with $\mu > 5''$ do not evince any galactic condensation, except perhaps for stars, fainter than $9^m.0$. The numbers with $\mu > 3''$ very clearly show a condensation for all magnitudes in the direction of the Milky Way, although to a less high degree than the numbers of stars with $\mu > 0''$.

TABLE 1. The numbers of stars of determined magnitude and proper motion for the whole sky.

m \ μ	0—2.9	3.0—4.9	5.0—7.9	8.0—9.9	10.0—14.9	15.0—19.9	20.0—29.9	30.0—49.9	> 49.9	Total
1—2.9	31	26	13	8	14	6	13	10	11	132
3.0—3.9	48	41	39	19	39	18	28	14	17	263
4.0—4.9	303	167	128	50	113	57	39	29	21	907
5.0—5.9	1090	665	575	222	329	140	117	72	52	3262
6.0—6.9	4019	2675	1528	564	728	349	185	123	82	10253
7.0—7.9	14664	8268	4150	1092	1716	530	374	281	125	31200
8.0—8.9	44996	27517	11992	2417	3718	930	744	464	186	92964
9.0—9.9	108257	78185	32478	8420	9142	2165	1203	480	241	240571
10.0—10.9	270748	172763	136019	29009	29653	2579	2579	645	645	644640
11.0—11.9	705391	436968	288711	74909	43697	4681	3121	1561	1561	1560600

This result agrees with that found by KAPTEYN in 1893 and with the conclusions, which DYSON and THACKERAY found lately from a discussion of the Greenwich 1910 catalogue.

The mean parallaxes $\pi_{m,\mu}$ for the whole sky and for the different galactic zones have not been determined by us directly. In our opinion they may be reduced for our investigation, preliminary as it is, with sufficient accuracy from former researches.

KAPTEYN found in *Publ. Groningen* N°. 8 for these mean parallaxes

$$\pi_{m,\mu} = a\,\mu^b\,\varepsilon^{m-5.0}$$

in which $a = 0''.0038$, $b = 0.71$ and $\varepsilon = 0.905$.

Use was made of the value 16.7 Kilometers a second for the velocity of the sun. If we accept the modern value, 19.5 Kilometers a second, the mean parallaxes π_m, used by KAPTEYN in deducing his formula, are lessened by 14%. If we now suppose — as VAN RHIJN also does [1]) — that this correction for the mean parallaxes π_m changes only the value of a in $\pi_{m,\mu}$, we find $a = 0''.0032^5$.

KAPTEYN's parallaxes corrected in this way agree with the values, deduced by Dr. VAN RHIJN in his dissertation in an investigation, based on other and more modern data than those used by KAPTEYN, in which he partly also applied another method. Hence it follows that the parallaxes $\pi_{m,\mu}$, as given in *Publ. Groningen* N°. 8, are very reliable.

In order to derive from the parallaxes which may be applied to the whole sky, the $\pi_{m,\mu}$ for the different zones, we have assumed, that the $\pi_{m,\mu}$ of the various galactic zones are related in the same way as the π_m of these zones.

This assumption is rather arbitrary. Most probably it cannot be proved to be correct in all respects. There is reason to believe, however, that the error, made in this way, is not very great. Therefore we thought we might use this hypothesis in our preliminary research. We have already mentioned that we were able to use, thanks to the great kindness of Professor KAPTEYN and Dr. VAN RHIJN, the mean parallaxes of stars of determined magnitude and galactic latitude, which will soon be published in *Publ. Groningen* N°. 29. These parallaxes may be respresented by the following formulae:

Zone	I	$\log.\ \pi_m = 8.883 - 0.142\ m$
	II	$= 8.904 - 0.142\ m$
	III	$= 8.957 - 0.142\ m$
	IV	$= 9.024 - 0.142\ m$
	V	$= 9.066 - 0.142\ m$
Whole sky		$= 8.943 - 0.142\ m$

[1]) Diss. Groningen 1915, p. 35.

We have calculated — if we call the parallaxes, used formerly by KAPTEYN, $\pi_{m\,\text{KAPTEYN}}$, and if for convenience' sake we indicate those meant above by $\pi_{m\,\text{VAN RHIJN}}$ — the $\pi_{m,\mu}$ for the whole sky and for the 5 zones by multiplying the numbers of table G of *Publ.*

TABLE 2.

The mean parallaxes of stars of determined magnitude and proper motion for the whole sky.

μ / m	1.5	4.0	6.5	9.0	12.5	17.5	25	40
2.6	0″.0065	0″.0132	0″.0187	0″.0237	0″.030	0″.037	0″.049	0″.068
3.6	.0057	.0115	.0163	.0206	.026	.033	.042	.059
4.6	.0049	.0100	.0142	.0178	.023	.029	.037	.051
5.6	.0043	.0087	.0123	.0155	.020	.025	.033	.045
6.6	.0037	.0076	.0107	.0135	.017	.022	.028	.039
7.6	.0033	.0066	.0093	.0117	.014	.019	.025	.034
8.6	.0029	.0057	.0081	.0103	.012	.016	.022	.029
9.6	.0025	.0050	.0071	.0090	.011	.014	.019	.026
10.6	.0022	.0043	.0061	.0077	.009	.012	.016	.023
11.6	.0019	.0036	.0051	.0065	.008	.010	.014	.020

Groningen N°. 8 by a factor $\frac{\pi_{m_{\text{VAN RHIJN}}}}{\pi_{m_{\text{KAPTEYN}}}}$. The parallaxes, which may be applied to the whole sky, are given in table 2.

We ourselves have not tried to determine once more the probable deviation of the error curve log. π/π_0. Moreover we have supposed that ϱ does not vary with the galactic latitude. We made use of the value 0.19, found by KAPTEYN both in our solution for the whole sky and in that for the 5 galactic zones. It has, however, been proved in *Publ. Groningen* N°. 11 that the value of ϱ has only little influence on the result.

3. *Applying* KAPTEYN's *Method.*

We have applied KAPTEYN's method without any modification. For a description in detail of this method we refer to his treatise in *Publ. Groningen* N°. 11, which we cited already above. We shall limit ourselves to a short discussion of the hypotheses that have been made and an explanation of the tables mentioned in this essay.

The hypotheses, made in KAPTEYN's investigations, are three in number:

1stly the density is only a function of r;

2ndly the luminosity curve is independent of the distance from the sun and there is no absorption of light in space;

3dly the quantities $z = \log. \pi/\pi_0$ are distributed according to the law of errors.

The first hypothesis is necessary if we want to derive frequency-functions that may be applied to the whole sky. We seek for mean values for the unknown quantities and so we cannot take into account the variations in the values with the galactic latitude and longitude. KAPTEYN's method may, however, be used just as well, if we want to reckon with the influence of the galactic latitude by making separate solutions for the different galactic zones.

The second hypothesis can hardly be dispensed with. If we have certainty that the frequency-function of absolute magnitudes is everywhere the same in space, KAPTEYN's method offers the means to examine, whether there is a perceptible extinction of light and, on the other hand, when we know that there is no absorption we can examine if the luminosity curve varies with the distance from the sun. As neither one thing nor the other, however, is certain, we are obliged for the time being to make the supposition in question.

If we establish the frequency-function for the different galactic

zones separately, this difficulty is diminished considerably. We can then compare the luminosity curves, found for the various zones. If now, it appears that these curves correspond, it is highly improbable that the distribution of luminosities varies with the distance to the sun. If the luminosity law depends on r and not on b, we should find a fan-shaped composition of the sidereal system, which is not inconceivable, but highly improbable all the same.

If we may assume that the distribution of the luminosity is independent of the distance to the sun, KAPTEYN's method enables us to determine the absorption, or at least to examine if it has any influence on the distribution of stars in space which we found.

The third hypothesis was, indeed, when KAPTEYN made it for the first time a rather arbitrary assumption, and it must be conceded that other results would have been found, if another hypothesis had been made. The hypothesis was made, because at that time there were still too few measured parallaxes, to enable us to deduce the form of the frequency curve $log.\ {}^{\pi}/_{\pi_0}$ directly from the data. Perhaps this will be possible when applying the method once more. It is, however, of little importance to discuss in the present stage the question whether this hypothesis could be justified or not, as SCHWARZSCHILD has proved that the above-mentioned relation exists for a special form of the density law and the luminosity law, which form obtained a great amount of probability owing also to his investigations.

We have deduced the luminosity law from the data of observation mentioned in § 2. Space was divided into a number of shells the radii of which had been selected for convenience' sake in such a way that $log.\ r$ increases with 0.2. Afterwards the mean parallax was determined for each of the numbers $N_{m,\mu}$ that had been found by means of the formula for $\pi_{m,\mu}$. Then with the aid of the value found for the probable deviation of the error curve $log.\ {}^{\pi}/_{\pi_0}$, it was calculated which part of the numbers found occur in every shell.

The numbers $N_{m,\mu}$ which we found for the whole sky are given in table 1, and the corresponding parallaxes $\pi_{m,\mu}$ in table 2. In table 3 we have mentioned, how these numbers of stars are divided over the various shells. Tables in accordance with this we have calculated for the five zones.

Now we have derived from these tables others, indicating the numbers of stars of every absolute magnitude per unit of volume in the different shells. For the whole sky we communicated our results in table 4. Between the fat-faced lines the numbers

TABLE 3. The number of parallaxes for the whole sky.

π	Mag. / π	2.6	3.6	4.6	5.6	6.6	7.6	8.6	9.6	10.6	11.6	Total
0″.00000—0″.00100		0	1	6	33	228	83	4325	17045	55568	212210	289499
.00100— .00158	0″.00118	1	2	20	94	469	3339	9806	28249	90900	245902	378782
.00158— .00251	.00187	3	6	55	247	1170	4603	15720	45170	112684	341132	520790
.00251— .00398	.00296	7	14	99	435	1781	6153	20643	52073	119250	328492	528947
.00398— .00631	.00469	14	25	147	579	1957	6700	18139	46380	139905	234121	447967
.00631— .0100	.00743	16	38	150	597	1780	4721	12972	28374	77002	133400	259050
.0100 — .0158	.0118	24	47	146	503	1430	3155	6896	15180	34288	43900	105569
.0158 — .0251	.0187	21	46	121	389	806	1030	2715	5702	11082	17370	39282
.0251 — .0398	.0296	15	26	65	171	358	678	1222	1752	2806	2247	9340
.0398 — .0631	.0469	11	28	56	117	147	553	267	347	465	202	2193
.0631 — .1000	.0743	6	10	17	27	37	47	62	53	45	47	351
.1000 — .158	.118	2	2	4	7	7	10	11	5	0	16	64
> .158	.204	1	1	0	1	1	3	0	0	0	0	7
		121	246	886	3200	10171	31075	92778	240330	643995	1559039	2581841

TABLE 4. The log. of the number of stars per unit of volume.

π \ M	−7.0	−6.0	−5.0	−4.0	−3.0	−2.0	−1.0	0.0	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	11.0	12.0
0″.00118	3.503	3.804	4.804	5.476	6.274	7.027	7.494	7.954	8.462	8.894										
.00187		4.580	4.881	5.843	6.496	7.171	7.766	8.299	8.758	9.145	9.636									
.00296			5.548	5.849	6.703	7.341	7.954	8.492	9.018	9.420	9.779	0.220								
.00469				6.449	6.701	7.470	8.066	8.595	9.129	9.562	9.969	0.449	0.672							
.00743					7.107	7.483	8.079	8.679	9.153	9.577	0.016	0.356	0.797	1.028						
.0118						7.883	8.175	8.667	9.205	9.658	0.002	0.342	0.684	1.038	1.145					
.0187							8.425	8.766	9.186	9.693	0.009	0.116	0.537	0.859	1.148	1.343				
.0296								8.879	9.118	9.516	9.936	0.257	0.534	0.790	0.947	1.151	1.055			
.0469									9.344	9.750	0.051	0.371	0.470	1.046	0.730	0.843	0.970	0.608		
.0743										9.681	9.903	0.133	0.334	0.471	0.575	0.695	0.627	0.556	0.575	
.118											9.804	0.804	0.105	0.348	0.348	0.503	0.544	0.202	—	0 707
.204												9.979	9.979	—	9.979	9.979	0.456	—	—	—

of stars of the apparent magnitudes 3.0 to 10.0 are found. These are most reliable.

The densities in the different shells were determined by comparing each time in two successive shells the numbers of stars of a determined absolute magnitude. The relative density of two successive shells was found as the average of three determinations, based respectively on the stars of the apparent magnitudes from 3.0 to 8.0, 3.0 to 9.0 and 3.0 to 10.0. The density in the fifth shell, *viz.* the one for which the mean parallax is 0".0296, was supposed to be *one*, after which the density at every distance from the sun, expressed in this unity, is known.

The average density for the whole sky varies about in the same way with the distance to the sun as KAPTEYN found. In zone I the density at great distances is considerably more than the average, in zones III, IV and V on the other hand it is much less.

By means of the densities found we now calculated from table 4 and the similar ones for the galactic zones the number of stars per unit of volume, after the density had been reduced to the one for $\pi = 0''.0296$. In order to do so each number in the last-mentioned tables was diminished by the logarithm of the density of the shell pertaining to it. In this way for instance table 5 has been calculated from table 4.

From the tables that were found last of all, the luminosity curves for the whole sky and the 5 zones may be deduced at once. The numbers, standing between the fat lines in each column, correspond pretty well.

In table 5 we took the averages of these numbers (not of the logarithms), and noted down the logarithm of these averages in the last line of each table. In taking the mean equal weight was ascribed to all numbers, except to those of the first four shells. These numbers are not very reliable, because they are small, but especially because we were obliged to exclude from our investigation stars with a proper motion $> 50''$ per century. Consequently the luminosity curves that have been found are only of value up to $M = 5.0$.

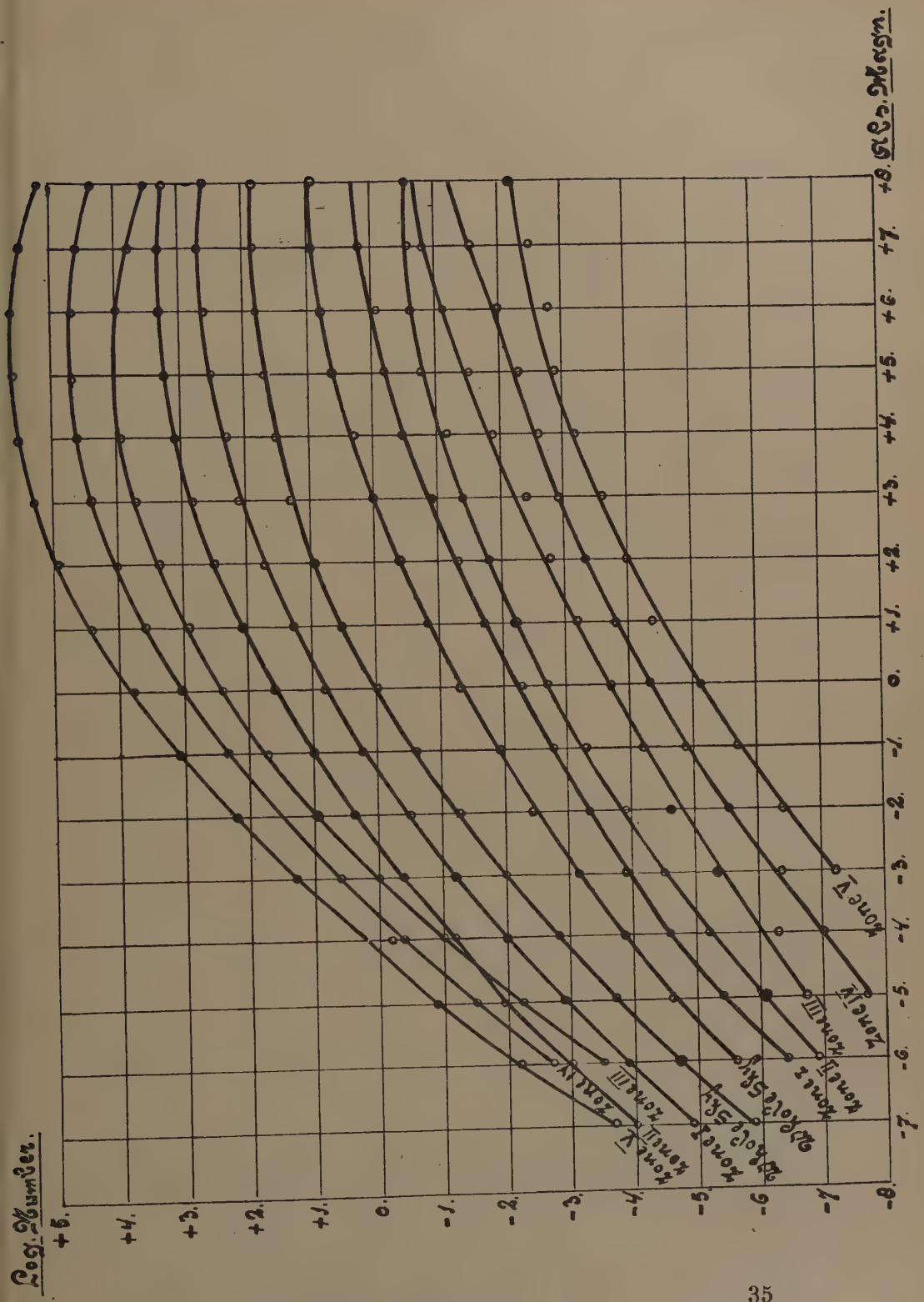
It is of interest to point out that our result for the whole sky corresponds beautifully to KAPTEYN's found in *Publ. Groningen* N°. 11.

Furthermore it is remarkable that the curves found for the various zones differ only a little. We already observed that from this we may conclude with a certain probability that the luminosity curve does not change with the distance to the sun either.

In table 5 and the corresponding tables for the 5 galactic zones

TABLE 5. The log. of the number of stars per unit of volume, reduced to $\pi = 0''.0296$.

π \ M	−7.0	−6.0	−5.0	−4.0	−3.0	−2.0	−1.0	0.0	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	11.0	12.0
0″.00118	4.141	4.442	5.442	6.114	6.912	7.665	8.132	8.592	9.100	9.532										
.00187		4.976	5.277	6.239	6.892	7.567	8.162	8.695	9.154	9.541	0.032									
.00296			5.741	6.042	6.896	7.534	8.147	8.685	9.211	9.613	9.972	0.413								
.00469				6.527	6.779	7.548	8.144	8.573	9.207	9.634	0.047	0.527	0.750							
.00743					7.153	7.529	8.125	8.725	9.159	9.623	0.062	0.402	0.843	1.074						
.0118						7.902	8.194	8.686	9.224	9.677	0.021	0.361	0.703	1.057	1.164					
.0187							8.426	8.767	9.187	9.694	0.010	0.117	0.538	0.860	1.149	1.344				
.0296								8.879	9.118	9.516	9.936	0.257	0.534	0.790	0.947	1.151	1.055			
.0469									9.269	9.675	9.976	0.296	0.395	0.971	0.655	0.768	0.895	0.533		
.0743										9.932	9.846	0.384	0.585	0.722	0.826	0.946	0.878	0.807	0.826	
.118											0.249	0.249	0.650	0.793	0.793	0.948	0.989	0.647	—	1.152
.204												0.712	0.712	—	0.712	0.712	1.189	—	—	—
Mean	4.141 :	4.442	5.367	6.140	6.873	7.572	8.146	8.679	9.182	9.629	0.017	0.299	0.600	0.827	0.947	0.870 :	1.038 :	0.647 :		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+4.
+3.
+2.
+1.
0.
-1.
-2.
-3.
-4.
-5.
-6.
-7.
-8.
Abs. Magn.
+8.
+7.
+6.
+5.
+4.
+3.
+2.
+1.
0.
-1.
-2.
-3.
-4.
-5.
-6.
-7.
Zone V
Zone IV
Zone III
Zone II
Zone I
Whole Sky
Zone I
Zone II
Zone III
Zone IV
Zone V
Zone IV
Zone III
Zone II
Whole Sky
Zone I
Zone II
Zone III
Zone IV
Zone V

the numbers in each column, as has been already observed, agree with each other. If our conclusion that the frequency curve of the absolute magnitudes does not vary with r, is right, we may derive from the agreement of the luminosity curves in the different shells that there is no perceptible absorption of light in space.

In figure 1 the frequency curves that were found have been drawn. The six lines in the lowest part of the figure refer to the determination of the luminosity curve, discussed in this essay. The six lines in the upper part refer to an application of SCHWARZSCHILD's method to the same data, which will be explained in a following communication.

The line representing our determination of the luminosity curve for the whole sky, indicates the logarithm of the number of stars of every M per unit of volume in the neighbourhood of the sun. For the other curves we added, in order to make comparison possible, a constant amount to each number.

Amsterdam, June 1918.

Mathematics. — "*On the arising of a precession-motion owing to the non-euclidian linear element of the space in the vicinity of the sun*". By Prof. J. A. SCHOUTEN. (Communicated by Prof. LORENTZ).

(Communicated in the meeting of June 29, 1918).

If k be an curve in an n dimensional space X_n of arbitrary form, there will be in the euclidian space of $\frac{n(n+1)}{2}$ dimensions, into which X_n can always be placed without changing its linear element, a euclidian space Y_n, i. e. a space Y_n developable on a plane space tangent to X_n along k. If in the euclidian space Y_n a system of n mutual $\perp$ directions be moved with its origin along k parallel to itself, we find that these directions in X_n define a "geodesically moving system" [1]). If two arbitrary spaces are tangent to each other in a curve k, it follows from this definition, that a system geodesically moving along k for one space, will geodesically move for the other space too. A volume-element covered with mass can move in X_n as a solid body, but for some infinitesimals of a higher order. If a suchlike element always remains at rest with regard to a geodesically co-moving system of directions we will call it *compass-body*. Hence the compassbody mechanically realizes the geodesically moving system.

If k be a closed curve, the initial position will as a rule not coincide with the final position, if X_n is non-euclidian. Thus the position of the compassbody is changed with every rotation. Now according to the investigations of K. SCHWARZSCHILD [2]) the space in the vicinity of the sun is not euclidian, but very slightly curved. The linear element is of the form

$$ds^2 = dR^2 + R^2\, d\theta^2 + R^2 \sin^2\theta\, dq^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

[1]) Cf. for a more detailed exposition of the geodesically moving system: "Die direkte Analysis der neueren Relativitätstheorie". Verh. of the Kon. Akad. v. Wet. Vol. 12. No. 6 and "On the number of degrees of freedom of the geodesically moving system and the enclosing euclidian space with the least possible number of dimensions". Proc. of the Kon. Akad. v. Wet. May 25, 1918.

[2]) Ueber das Gravitationsfeld eines Massenpunktes nach der Einstein'schen Theorie, Berl. Sitzungsber. 1916, p. 189—196.

L. FLAMM [1]) has indicated how a space with that linear element can be realized. The parabola situated in the xz plane:

$$z^2 = 4a(x-a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

has the linear element:

$$ds = \frac{dx}{\sqrt{1-\frac{a}{x}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If this parabola is rotated about the z axis, in the xyz space, there arises a rotation-surface with the linear element:

$$ds^2 = \frac{dR^2}{1-\frac{a}{R}} + R^2\,d\varphi^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

φ being the rotation-angle, measured from the yz plane (fig. 1).

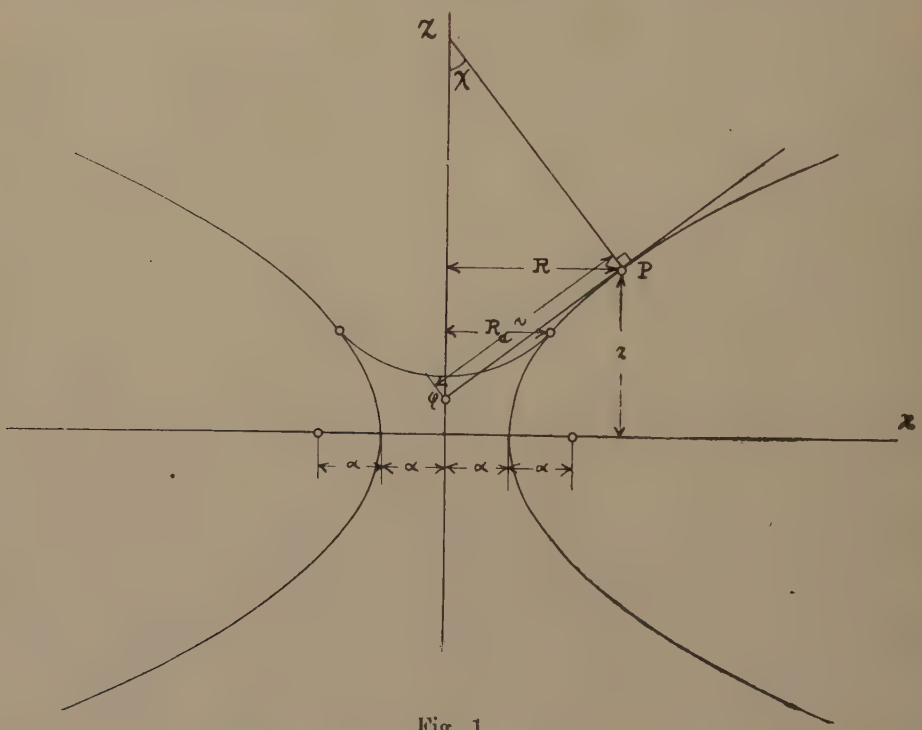

Fig. 1.

A fan of directions in the centre of the sun determines a system of ∞^1 geodesic lines, together forming a diametral surface. Such a diametral surface can as regards the parts beyond the surface of the sun be developed on the rotation-surface (4) without changing its linear element. R may approximately be equalized with the

[1]) Beiträge zur Einstein'schen Gravitationstheorie 17 (16) 448—454.

naturally measured distance relative to the centre of the sun, while $\alpha = 2.945.10^{15}$ cm. The circle described by a definite point P:

$$\left.\begin{aligned} x^2 + y^2 &= R^2 \\ z &= 2\sqrt{\alpha(R-\alpha)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

consequently represents a circle in the diametral surface, having the same centre as the sun. If the sun be looked upon as a globe, filled with an incompressible liquid, a diametral surface within the sun will have the same linear element as the globe-surface, which touches the described rotation-surface in a parallel-circle with a radius R_a. This radius R_a too may approximately be equalized with the astronomically measured radius of the sun. If the described rotation-surface (4) is rotated in the four-dimensional $xyzu$ space around the yz plane, there arises a curved three-dimensional space with the linear element (1) when θ is the angle of rotation, measured from the yzu space.

We shall now investigate the motion of a compassbody moving in the circle (5) around the sun. For this purpose it is sufficient to find a space, tangent to (1) in (5), and in which the geodesic motion can conveniently be indicated. We now make the tangent line PQ rotate with the parabola. That tangent line describes a cone with the linear element:

$$ds^2 = \frac{dR^2}{\cos^2\chi} + R^2 d\varphi^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in which equation χ only depends on the definitely selected point P, and therefore is a constant. With the second rotation there arises from this cone a space with a linear element:

$$ds^2 = \frac{dR^2}{\cos^2\chi} + R^2 d\theta^2 + R^2 \sin^2\theta\, d\varphi^2 \quad . \quad . \quad . \quad . \quad . \quad (7)$$

in which χ is once more a constant. The linear element of a euclidian space may be expressed (in polar-coordinates) R', φ', θ':

$$ds^2 = dR'^2 + R'^2 d\theta'^2 + R'^2 \sin^2\theta' d\varphi'^2 \quad . \quad . \quad . \quad . \quad (8)$$

and by the substitution:

$$\left.\begin{aligned} R &= R' \cos\chi \\ \varphi &= \frac{\varphi'}{\cos\chi} \\ \theta &= \frac{\theta'}{\cos\chi} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

(7) passes into:

$$ds^2 = dR'^2 + R'^2 d\theta'^2 + R'^2 \sin^2\frac{\theta'}{\cos\chi} d\varphi'^2 \quad . \quad . \quad . \quad (10)$$

On the curve (5) we have $\cos\theta = 0$. With the exception of quantities of the order χ^4 the tangent space (7) behaves along the curve (5) as the euclidian space (8). Hence we need only trace the movement of the geodesically moving system in (8) along the curve corresponding to (5). As the coordinates φ and θ have obtained a factor $\cos\chi$ according to (9), (8) can be realized by the part of the euclidian space xyu that remains, when a rotation-cone having for axis the y-axis and a top-angle of $2\pi\ (1-\cos\chi)$ is taken away. (fig. 2).

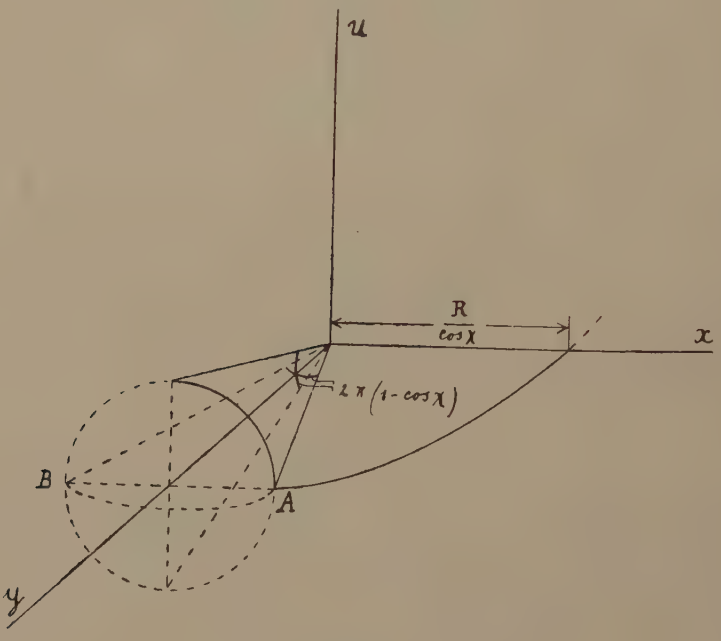

Fig. 2.

The part of the circle:

$$\left.\begin{aligned} x^2 + y^2 &= \frac{R^2}{\cos^2\chi} \\ u &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (11)$$

situated within this space viz. the part extending in fig. 2 from A via a point in the negative part of the y-axis to B, will then correspond with the entire curve (5).

This result may also be obtained by replacing the cone (6) by its unfolded mantle, laid down in the yx plane symmetrically to the y axis, being a sector with an angle $2\pi\cos\chi$. The curve (5) will then coincide with (11). During the revolution about the yz

plane the circle-sector describes the space-section described. By this method it does not at once become obvious that now the obtained euclidian space may indeed replace the originally existent non-euclidian tangent-space along (5). The motion of the compass-body can now be easily traced. In the euclidian space xyu the compassbody always moves parallel to itself. If a constant direction in A in this body has the direction of the radius, then that direction in B forms with the radius an angle $2\pi(1 - \cos\chi)$ situated in a plane // at the xy-plane. Now $\frac{a}{R}$ is very small, hence:

$$\cos\chi = 1 - \frac{a}{2R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

so that the total deviation δ in one revolution amounts to:

$$\delta = \pi\frac{a}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

A compassbody moving around the sun, as its central point, in a circle with a radius equal to the average distance from the earth to the sun will show according to this formula after one revolution a deviation of 0.013″. If the radius is equal to the average distance from Mercury to the sun, the deviation amounts to 0.0328″. If the radius is equal to the radius of the sun, the deviation amounts to 2.73″. If, from another cause, the compass-body already has a revolution around an axis, which is oblique relatively to the plane of the orbit, there will set in, merely on account of the deviation described, a precession-motion, which in the first of the above-mentioned cases would result in a complete revolution of the equinoxes after ± 100.000.000 years. It is noteworthy that the effect described is of the same order as the deviation of a ray, passing the sun at a distance R from the central point. According to Einstein this deviation indeed amounts approximately to $\frac{2a}{R}$.

Whether the deviation computed of the precession motion for the earth will indeed set in, depends on the question to what extent and to what approximation a mass of the quantity and the composition of the earth has the proporties of a compassbody. In order to answer this question it is necessary to make definite suppositions as regards the physical qualities of the earth, in particular the mutual attraction of her parts, and starting from these suppositions, to integrate the four-dimensional dynamical equations of motion.

ADDENDUM.

Prof. DE SITTER, to whom I have communicated the above, writes: A precession of 0".013 per annum of course comes well within the reach of observations, since the observed value of the precessional constant is trustworthy to about 0".0010. The point is therefore with what accuracy the theoretical value can be computed. Now the lunisolar precession (the planetary precession can be taken as completely known) is given by a formula of the form

$$p_1 = (P + Q\mu).H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where P and Q are known numbers and μ is the mass of the moon (expressed in that of earth + moon as unit) and $H = \frac{C-A}{C}$ depends on the moments of inertia of the earth. The uncertainty of μ causes in p_1 an uncertainty of about $^1/_{2000}$ of its amount, thus, if H was exactly known, p_1 would be uncertain to the extent of $\pm$ 0",025 or twice the new precession. A better determination of μ may be expected from the opposition of EROS in 1931 [1]). However the uncertainty of H is of much greater importance. In 1915 [2]) I have with the aid of the hypothesis of isostasy, derived the ellipticity ε of the earth from H, this latter being determined from p_1 by (1). To invert this order it would be necessary, in order to get a p.e. of $\pm$ 0".005 in p_1, to know ε to about $^1/_{20000}$ of its amount. The direct determinations of ε at present do not go beyond about $^1/_{300}$. To increase this accuracy seventyfold would in my opinion be beyond the forces of geodetical science, at least in the near future.

We can determine H with greater accuracy from the constant of nutation, which is given by

$$N = R\,.\,\mu\,.\,H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where R is again a known number. From (1) and (2) we derive

$$p_1 = \left(S + \frac{T}{\mu}\right).N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where S and T are also practically exactly known. The uncertainty of the multiplier owing to the uncertainty of μ is now about $^1/_{4000}$ or $^1/_{5000}$, and it will probably be reduced to $^1/_{10000}$ by the new determination of μ in 1931.

[1]) The figure of the earth and some related astronomical constants. The Observatory Aug. 1915, page 322.

[2]) On Isostasy, the moments of inertia, and the compression of the earth. These Proceedings April 1915, Vol XVII, p. 1291.

The result of NEWCOMB's discussion in 1891 of all available determinations of the constant of nutation has determined its value to about $^1/_{11000}$ of its amount. To get a p.e. of $\pm$ 0".005 in p this accuracy must be increased nine- or tenfold. This certainly is no easy task, but it would be preposterous to say that it exceeded the forces of astronomy. It will of course require very refined and prolonged observations and discussions.

I may be allowed to remark that it still remains to be investigated whether the new precession is the only effect of the new gravitational theory, and the equations (1) and (2) are not affected, i.e. whether EINSTEIN's theory gives exactly the same equations for the motion of the axis of rotation of the earth with reference to the geodetically transported system of coordinates, as are found in NEWTON's theory relatively to a "fixed" system. This cannot be asserted without a special investigation, which so far as I know has not been undertaken, and it might even happen that the precession of the geodetically transported system of coordinates was exactly cancelled by a small change in the precession of the earth relatively to that system.

Physiology. — *"Effects of the Rays of Radium on the Oögenesis of Daphnia pulex"*. By Miss M. A. VAN HERWERDEN. M. D. (Communicated by Prof. C. A. PEKELHARING.)

(Communicated in the meeting of Sept. 29, 1918).

For a considerable time I have been prosecuting the effects of radium radiation on a race of Daphnia pulex bred in the laboratory, with whose method of reproduction I had become thoroughly acquainted during a study of rather more than 8 years. My original purpose was to evolve, if possible, parthenogenesis in the sexual period, or conversely, to impart to the parthenogenetic females the faculty of producing a sexual offspring. The importance of this experimentation was sufficiently borne in upon me in connection with the view adopted by many researchers that by radiating the organism with radium, enzymic actions are accelerated or diverted. A short or a prolonged radiation with 0,7 mgrs or 3,1 mgrs of radiumbromide, at my disposal, never resulted in any effect upon the sexual or the parthenogenetic stage, whereas after a radiation with a stronger preparation the animals succumb. [1]) Nevertheless I considered it a point of importance to continue the experiments, since they throw a peculiar light upon the resistance of the protoplasm in the several phases of the oögenesis and of the embryonic development under the influence of radium rays.

Daphnia pulex affords extremely fit material for such experiments. Besides being fairly transparent and easy to watch under the microscope, it also enables us not only to follow in the living animal the development of the parthenogenetic eggs located in the broodpouch, but also to determine the degree of maturation of the eggs in the ovary. Sometimes the amount of yolk in the maturing eggs enables us to foretell correctly to a few hours, when they will leave the ovary. There is always plenty of material for control, as several young ones can be expected at every parturition. Over and above, the rapid succession of generations grants a comprehensive survey not only of the animals under examination, but also of a large progeny.

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[1]) Verslag "Koninklijke Akad. v. Wetenschappen" Deel XX p. 20.

In the experiment I placed Daphnia pulex, embedded in a drop of ditchwater, immediately on the micaplate of the radium-preparation. In the following descriptions I shall designate the preparations used, containing 0,7 and 3,1 mgrs. of radium-bromide respectively as capsule A and capsule B.

Animals belonging to different age-periods were radiated separately. It immediately appeared that the adult Daphnia is much less responsive to the radium rays than the new-born animal. An exposure of 18 hours on capsule A did not affect the animals so as to kill them. If the animal kept alive after this prolonged radiation, it was sometimes seen to succumb later on at the periodical ekdysis of the chitin shell, which was often attended by an abnormal chitin-formation. This then pointed to some injury to the ectoderm. Still, even when the animal remained perfectly healthy, it had become sterile for the rest of its life. In the case of only a few hours radiation without any yolk-rich mature eggs being noticeable in the ovary the reproductive faculty is not interfered with: the eggs leaving the ovary later on develop normally and the young generating from these eggs reach sexual maturity in the regular way and produce a healthy offspring.

However, when radiation takes place, while the ovary bears large yolk-rich eggs, or when the eggs have only just entered the broodpouch, a radiation of 25 *minutes on capsule A, or of some minutes on capsule B, will suffice to cause the eggs to develop abnormally,* so that they are destroyed already in the blastula-stage and are resorbed in the mother. Susceptibility varies individually, without any apparent connection with the age of the mature females. Whereas the eggs that have only just entered the broodpouch are as susceptible as, or more susceptible than before leaving the ovary (abortus sometimes occurs already after $1^1/_2$ minutes' sojourn on capsule B), the resisting power of the embryos is seen to increase during the development. Viable young were developed even after a three hours' radiation on capsule A in the gastrula-stage. Yet in these cases the brood did not seem to be always in good condition, since one of the young emerging from such a radiated gastrula — the only one of this lot that reached sexual maturity — produced a brood of an anomalous morphological structure. It appears then, that the future germ-cells of the gastrula in this case must have been injured already during the radiation.

A greater resisting power is displayed by the almost viable young, as they can stand a 20 hours' radiation on capsule *A*, without any prejudice to their future fecundity. If, however, these young ones

leave the brood-pouch during the radiation, they invariably perish, every one of them; a sojourn on capsule *A* of a couple of hours only will kill them. After an hour's radiation the cardiac action is weakened and irregular and they die soon after. In the brood-pouch they are presumably walled off by the chitin shell of the mother and by the fluid in which they are swimming, which protection, however, is not sufficient for the young embryos, which are so much more susceptible to the radium rays.

A short radiation of a female with maturing egg-cells in the ovary, leads indeed to destruction of the eggs; but it leaves the mother unhurt. Afterwards it even causes sometimes a more numerous offspring, which phenomenon is analogous to that seen in the action of various poisons on Daphnia pulex [1]), on which a small dosage of the poison acts as a stimulus. The resistance of eggs from one and the same lot sometimes differs very much, as among the eggs that are being destroyed occasionally a single normal young may be seen that does not appear to have suffered at all from the noxious influences that threatened it before the embryonic stage, and later on may possibly possess a normal faculty of reproduction. An anomaly in the structure of such a young occurs only rarely. It cannot be said to be typical for radium-radiation. Generally these monstra (with abnormal profile or defective intestine) are few and far between. As a rule, therefore, radiation yields a normal embryonic development or none at all. This is the reason why I never succeeded yet in breeding mutations of Daphnia pulex by radiation with radium, as MORGAN achieved on a large scale with the Drosophila fly. The few abnormal specimens never reached maturity, one excepted, which recovered completely and produced a normal offspring that was still healthy after four months. Indeed my eight years' experience with Daphnia pulex have convinced me that this race though highly modifiable, shows only slight mutability.

Other researches in experimental embryology also show that eggs from the same lot vary as to their resisting power to noxious effects. I here call attention to the researches of PEARL [2]) on the difference in degree of resistance evinced by embryos of the domestic fowl to intoxication with alcohol, in which case also the eggs, liable to reach full development bring forth normal chickens.

It might be objected that in the radiation experiments the eggs are differently exposed. This may occur with a numerous brood, in which the inner eggs are shielded from the noxious rays by the

[1]) l.c. p. 1.

[2]) Proceedings of the National Acad. of sciences. U. S. A., Vol. II, p. 380.

external eggs. But the exposure will presumably be the same for the Daphnia moving about freely in the waterdrop on the radium capsule, when, as in our case, there are only 4 or 5 eggs located in the brood-pouch.

What is the cause of this unequal resistance of the Daphnia eggs? A reduction-division of the chromosomes, which might be responsible for the unequal distribution of paternal and maternal hereditary units does not occur in this parthenogenetic development; the egg retains the number of chromosomes of the mother. A similar difference in the reaction of the eggs to the noxious influences I previously detected, when treating the Daphniae with phenylurethan [1]).

Whereas with a short radiation of a female with mature egg-cells only the first brood succumbs, and the succeeding broods are normal (even very abundant), with a longer radiation also the eggs in a younger stage of development are seen to be damaged, until ultimately the Daphnia becomes completely sterile. After one single radiation a Daphnia may, after many abortus, produce quite unexpectedly normal young again, if namely the younger oögonia are not damaged. Such an after-effect I observed up to the 6th of January 1917 of a radiation for some hours on capsule *A* on the 23d of November 1916. After the first date again young were born that were completely normal.

My experience that eggs of Daphnia pulex in the last stage of maturation are most susceptible to radium rays, and that only after a prolonged radiation also the younger egg-cells and at last the oögonia are injured, accounts for PACKARD's [2]) experience that after radiation of the Drosophila larvae, the young flies become sterile for some weeks, and afterwards become fertile again. If we bear in mind that results with mammals also favour the theory that especially the mature egg-cells are very susceptible, we are justified in presuming that this holds for the whole animal kingdom.

How to account for the fact that the maturing eggs are more susceptible to the radium rays than the immature and the remaining cells of embryonic and maternal organism? Again, what molecular transformations occur in the protoplasm under the influence of radium radiation? The view adopted by several researchers that enzymic actions are accelerated or diverted, prompted me to compare the embryonic development of radiated and non-radiated sisters, which as to temperature and diet had been bred under the same conditions.

[1]) l.c. page 1.

[2]) Journal of exp. Zoology. XIX, p. 332.

Up to the moment the eggs left the ovary these sisters were kept in the same culture glass, animals being selected for this experiment whose eggs entered the brood-pouch in the same hour. As soon as the eggs began to develop one Daphnia was radiated for fifteen minutes on capsule *A*.

After this the conditions were made equal for either animal. In case a normal brood was developed after this radiation, embryogeny was neither retarded, nor accelerated. At the same hour the heart's pulsation became visible in the brood of the two animals; at the same time the development of the limbs commenced and the first eye-pigment developed itself; in the same hour the young left the parental organism. A similar observation was made at a second radiation. If only the radium action does not pass the physiological boundary there is neither acceleration nor retardation of development.

Is it the alpha-, the beta-, or the gamma-rays to which the egg-cells of the Daphnia are particularly responsive. In radiating on the capsule with radium-bromide the alpha-rays are screened out by the mica-plate of the capsule, which they cannot penetrate and consequently they do not reach the animal. When separating the Daphnia from the radium-preparation by a leaden platelet of 3 m.m. thickness, the beta-rays do not reach the Daphnia, while the secondary beta-rays are allowed to resorb through a mica-platelet of 50 μ thickness on which the Daphnia is placed. In this way Daphniae with maturing eggs in the ovary could stand a radiation with the gamma-rays from 0,7 mgrs. of radium-bromide for 24 hours, without abortus, which proves the harmlessness of the gamma-rays. When applying the radiumpreparation of 3,1 mgr., which, as has been seen, will destroy the eggs within a few minutes, a radiation of 24 hours with the exclusion of the beta-rays, could be borne without deleterious influences. In a few of the latter experiments, however, the first brood is aborted. It is, therefore, possible that to a stronger concentration of the gamma-rays (not obtainable with these preparations) the Daphnia egg-cells prove to be sensitive, a sensitivity, however, that is not to be compared with that to the beta-rays.

The antagonistic action between uranium and radium, demonstrated by ZWAARDEMAKER[1]) for the frog's heart, induced me to radiate Daphnia pulex in a drop of uranylnitrate and to determine whether resorption of the eggs stayed away in this process. A concentration of 600 mgrs. of uranylnitrate pro L. is tolerated for some hours without inhibiting the development of the brood; with

[1]) These Proc. XIX p. 1043.

a higher concentration the Daphnia itself succumbs in less time. When a Daphnia with maturing eggs in the ovary is placed in a solution of 500 mgrms of uranylnitrate pro L. and after half an hour again in a drop of the same fluid on the radium capsule of 3,1 mgrs, the radiation may be continued for ½ to 4 hours in a series of experiments without causing the brood to abort, while, under the same circumstances, the eggs, when placed in water, are fatally injured already after some minutes. Sometimes, however, the protecting influence of uranylnitrate was not at all discernible. Up to the present I have not been in a position to account for these various results. This also applies to the lower concentration of uranylnitrate.

More than fifty Daphniae were examined in microscopic sections and compared with non-radiated specimens. *Normal* maturation of eggs in Daphniae has already been described by KÜHN [1]). Broadly speaking my findings for the normal eggs are in agreement with his. A prolonged radiation did not enable me to detect in the maturing eggs any change either in the chromosomes, or in the nuclear body, or in the egg-plasma. Only in one polar spindle (for the formation of the first polar body) was the number of chromatin rods larger than could be anticipated with twice the number of chromosomes. Any possible alteration in the shape of the chromosomes is difficult to detect owing to the small dimensions.

Not before the blastula-stage, that is about the time when also in the living animal under the low-power microscope the embryos are seen to succumb, well-marked alterations take place in the nuclei, characterised by a collapse of the chromatin into coarse granules. The injury to the eggs, however, has been done long before the aided eye can detect it.

Though microscopic examination did not put us in a position to ascertain whether the noxious action of radium-rays has initially affected the nucleus, the cell-plasma or both, the high degree of susceptibility of the egg in a period when also considerable evolutions take place in the nucleus (formation of the polar spindle and decomposition of the large nuclear body) is indicative of a noxious effect of the beta-rays, especially on the nucleus. The fact that the first cleavage proceeds regularly and only at the close of it degeneration manifests itself, may be explained, when we call to mind BOVERI's [2]) investigations, which demonstrated that with the Sea-

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[1]) Arch. f. Zellforschung, Vol. I, p. 538.

[2]) Jenaische Zeitschr. Vol. 43, 1907.

urchin it is only at the close of the blastula-stage that the various properties of the chromosomes manifest themselves.

The period of maturation, in which the egg is so extremely susceptible to the radium rays, also proves to be the critical period for a poison as phenylurethan in a certain concentration, as discussed by me in an earlier paper [1]).

Just as some eggs of the brood sometimes escape death after radiation with radium, and develop into perfectly normal young, a Daphnia was occasionally developed after treatment with $\frac{1}{12000}$ n phenylurethan. When transmitted to water it produced a normal offspring. This again proves that the resisting power of the eggs to the danger, threatening them from the outer world, was occasionally very different, even with these parthenogenetic animals. But if they succumb in the struggle, the method of reaction in the two series of experiments is widely different. Whereas with a treatment with radium radiation this reaction leads irrevocably to degeneration at the close of the blastula-period, a treatment with phenylurethan evolves fully developed monstra, in consequence of a deleterious influence, exerted in the same period of susceptibility. These monstra, however, are not viable after birth; they are not resorbed, but are expelled from the parental organism.

Summary.

The egg-cells of Daphnia-pulex are most susceptible to radium radiation in the last stage of maturation. The resisting power increases in the embryonic stage.

In one and the same brood individual differences of susceptibility to the rays of radium is frequently noted. The egg that resists the deleterious influence often develops into a perfectly normal animal, which itself becomes fertile. The rare samples with morphological abnomalies seldom become adults. Only once did we succeed in breeding from such an abnormal young a stock without morphological anomalies.

A long-continued radiation from 0,7 mgrs of radium-bromide does not endanger the life of the sexually mature Daphnia, but only its fertility. It depends on the duration of radiation whether only the maturing eggs, the oöcytes, or also the oögonia are injured. Large progenies being easy of observation afford an opportunity to study this in every special case.

[1]) L. c. p. 1.

Prior to maturity Daphniae resist radium radiation for a long time. Only after a sojourn of many hours on the capsule with 0,7 mgrs of radium bromide the future ripening of the oögonia is also endangered.

A microscopic examination of the ovary and the embryos reveals that the deleterious effect of radium manifests itself only towards the close of the blastula-stage by an abnormal behaviour of the chromatin, also when the egg-cells were affected when lying still in the ovary.

If the beta-rays are eliminated through filtration the deleterious effect of radium is arrested or highly diminished, which proves the beta-rays to be mainly responsible for the destruction of the eggs.

Physiology. — "*The conduct of the kidneys towards some isomeric sugars (Glucose, Fructose, Galactose, Mannose and Saccharose, Maltose, Lactose).*" By Prof. H. J. HAMBURGER and Dr. R. BRINKMAN.

(Communicated in the meeting of September 28, 1918).

It has been proved by former researches [1]), that the glomerulus epithelium of the kidney of the frog is able to hold back glucose if the solution which is passed through the vascular system has a suitable composition. If one passes through the arteria renalis of the frog the following Ringer's solution: NaCl 0,7 %, KCl 0,01 %, $CaCl_2$ 0,0075 %, $NaHCO_3$ 0,02 %, in which 0,1 % glucose has been dissolved, then an artificial urine is excreted containing 0,07 % glucose; 0,03 % glucose has thus been retained by the glomerulus epithelium. If however the Ringer's solution contains 0,285 %, i.e. a quantity that corresponds to the titrational alkalicity of the serum of the frog, then much more sugar than 0,03 % is held back and not seldom the urine is free from sugar.

This phenomenon proves that the glomerus membrane, which is permeable to salts, is under physiological conditions impermeable to the also crystalline glucose.

In order to come to an explanation of this remarkable and useful contrast it seemed interesting to investigate how the glomerulus membrane would behave towards laevulose, galactose and mannose, all isomeric to glucose, and also towards the mutually isomeric saccharose, lactose and mannose.

Let us begin with the four first-named.

As is well known, the structural formula of the monosaccharides ($C_6H_{12}O_6$) can be represented in the following way:

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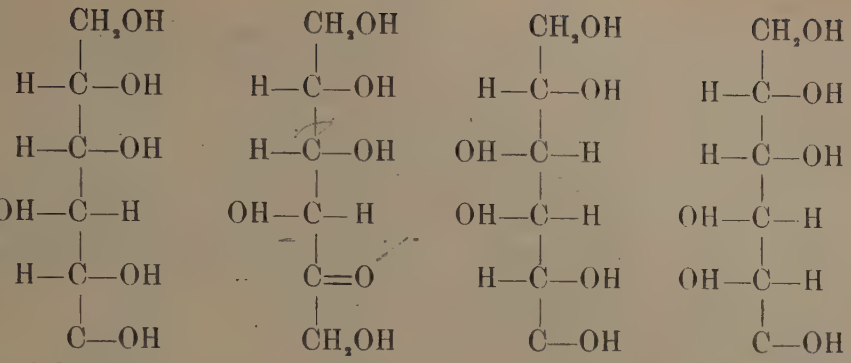
[1]) HAMBURGER and BRINKMAN: Proceedings of the Royal Acad. of Sciences Section of Jan. 27, and Sept. 29, 1917. Also: Biochem. Zeitschr. **88**, 97, 1918.



The experiments were made in exactly the same way as was described in the articles cited above. The perfusion liquid was of the following composition: NaCl 0,5 %, $NaHCO_3$ 0,285 %, KCl 0,01 %, $CaCl_2$ 0,2 %. This solution was prepared by mixing

50 ccm. NaCl 10 %,
50 ccm. $NaHCO_3$ 5,7 %,
10 ccm. KCl 1 % and
40 ccm. $CaCl_2$ 5 %

and adding boiled distilled water up to 1 Liter. Certain quantities of the sugars [1]) were dissolved in this solution, but still the reductive capability was estimated before each experiment. This was done in view of the possible errors in weighing or unknown differences in the amount of water contained in the sugars. BANG's method (1916) was used for estimating the reductive capability of the perfusion liquid as well as that of the urine excreted. The reductive power of the various sugars was *expressed in the percentage of glucose contained.*

As is well know the final titration is an estimation of Iodine with the aid of amylum; the amount of glucose contained is then computed from the quantity of Iodine necessary, by means of the formula $(a—0{,}12) : 4$, in which "a" is the number of *c.c.* solution of Iodine used.

A. Laevulose (Fructose).

Experiment 1 (July 11, 1918)

The perfusion liquid contains 0,1 % laevulose.

The reduction, expressed in glucose, amounts to 0,21 %.

0,1 ccm. urine from the right kidney needs 0,87 ccm. Iodine solution, which corresponds to $\frac{0{,}87—0{,}12}{4} = 0{,}0875\%$ glucose.

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[1]) We are indebted to Jhr. W. ALBERDA VAN EKENSTEIN, Director of the laboratory of the ministry of Finance and by Prof. H. J. BACKER, for several of the sugars.

0,1 ccm. urine from the left kidney needs 0,88 ccm. Iodine solution which corresponds to $\frac{0{,}88-0{,}12}{4} = 0{,}19\%$ glucose.

Retained by the right kidney $0{,}21-0{,}1875 = \mathbf{0{,}0225\%}$.
Retained by the left kidney $0{,}21-0{,}19 = \mathbf{0{,}02\%}$.

These quantities are so small that one may say that practically all the laevulose is allowed to pass by the glomerulus epithelium. The following experiments affirm this result.

Experiment 2 (July 12).

The same solution as used in experiment 1.
Reduction (0,1 c.c.) urine of right kidney $0{,}2125\%$.
Reduction (0,1 c.c.) urine of left kidney $0{,}215\%$.
Retained by right kidney $0{,}21-0{,}2125 = \mathbf{0}$.
Retained by left kidney $0{,}21-0{,}215 = \mathbf{0}$.

Result: *No* laevulose retained by the glomerulus epithelium.

Experiment 3 (July 13).

Experiments 1 and 2 were repeated with fresh solution.
Reduction 0,1 c.c. of solution passed $0{,}18\%$.
Reduction 0,1 c.c. urine of right kidney $0{,}18\%$.
Reduction 0,1 c.c. urine of left kidney $0{,}18\%$,

Result: *No* laevulose retained.

Experiment 4 (July 14).

The same solution passed as in experiment 3.
Reduction 0,1 c.c. urine of right kidney $0{,}1825\%$.
Reduction 0,1 c.c. urine of left kidney $0{,}1825\%$.

Result: *No* laevulose retained.

It has probably not escaped attention that the laevulose solution used above causes about twice as strong a reduction as a glucose solution of 0.1% viz. 0.18% on an average. Where it could be possible that a solution with a so much larger reductive capability could, just in connection with that fact, be allowed to pass by the glomerulus epithelium, we experimented with a solution in which there was 0.05% laevulose instead of 0.1%.

Experiment 5 (July 15).

RINGER's solution in which $0{,}05\%$ laevulose has been dissolved.
Reduction 0,1 c.c. of perfusion liquid $0{,}095\%$.
0,1 c.c. urine of right kidney: 0,5 c.c. Iodine solution; reduction $0{,}095\%$.
0,1 c.c. urine of left kidney: 0,52 c.c. Iodine solution; reduction $0{,}1\%$.
0,1 c.c. urine of right kidney: 0,52 c.c. Iodine solution; reduction $0{,}1\%$.
Retained by right kidney $0{,}095-0{,}095 = \mathbf{0}$.
Retained by left kidney $0{,}095-0{,}1 = \mathbf{0}$.
Retained by right kidney $0{,}095-0{,}1 = \mathbf{0}$.

Result: The diluted laevulose solution also passes completely through the glomerulus epithelium.

It seemed interesting to investigate whether laevulose could influence the retention of glucose.

In order to ascertain whether the circumstances, as regards the manner in which we had formerly worked with glucose, had indeed remained unaltered, several experiments were made with glucose alone.

B. Glucose and a mixture of Glucose and Laevulose.

Experiment 6. (July 18).

Perfusion liquid in which there was 0,1% glucose.
0,1 c.c. of solution, reduction 0,0975%.
0.1 c.c. urine of right kidney: 0.3 c.c. Iodine solution; reduction 0,045%.
0,1 c.c. urine of left kidney: 0,31 c.c. Iodine solution; reduction 0.0475%.
Retained by right kidney: 0,0975—0,045 = **0,0525%**.
Retained by left kidney: 0,0975—0,0475 = **0,05%**.

Experiment 7 (July 18).

The same perfusion liquid as in experiment 6.
0,1 c.c. urine of right kidney: 0,26 c.c. Iodine solution; reduction 0,035%.
0,1 c.c. urine of left kidney: 0,29 c.c. Iodine solution; reduction 0,0425%.
Retained by right kidney: 0,0975—0,035 = **0,0625%**.
Retained by left kidney: 0,0975—0,0425 = **0,0550%**.

Experiment 8 (July 20).

Perfusion liquid contains 0,07% glucose.
Reduction by 0,1 c.c. of solution passed 0,065%
0,1 c.c. urine of right kidney: 0,22 c.c. Iodine solution; reduction 0,025%.
0,1 c.c. urine of left kidney: 0,22 c.c. Iodine solution; reduction 0,025%.
Retained by each kidney: 0,065—0,025 = **0,04%**.

Experiment 9 (Sept. 20).

At the same time an experiment was made in which the perfusion liquid contained 0,2% glucose.
Reduction 0,1 c.c. of this solution 0,22%.
Reduction by 0,1 c.c. urine of right kidney 0,095%.
Reduction by 0,1 c.c. urine of left kidney 0,1125%.
Retained by right kidney: 0,22—0,095 = **0,1250%**.
Retained by left kidney: 0,22—0,1125 = **0,1075%**.

From these experiments it follows that now, as formerly, a quantity of glucose is retained, which is physiologically present in the blood of the frog. A remarkable contrast thus exists between the permeability of the kidney to glucose and to laevulose.

With this marked difference it appeared to be of importance to ascertain whether laevulose was perhaps capable of altering the permeability to glucose.

Both these substances were therefore dissolved in the perfusion liquid.

Experiment 10 (Sept. 20).

The perfusion liquid contains 0,1% glucose and 0,05% laevulose.
Reduction 0,1 c.c. of solution passed 0,205% .
0,1 c.c. urine of right kidney: 0,58 c.c. Iodine solution; reduction 0,1175%.
0,1 c.c. urine of left kidney: 0,60 c.c. Iodine solution: reduction 0,12%.
Retained by right kidney: 0,205—0,1175 = **0.0875%**.
Retained by left kidney: 0,205—0,12 = **0,085%**.

Experiment 11 (Sept. 20).

The same perfusion liquid as in experiment 9.
Reduction 0,1 c.c. urine of right kidney 0,155%.
Reduction 0.1 c.c. urine of left kidney 0,1575%.
Retained by right kidney: 0,205—0,155 = **0,05%**.
Retained by left kidney: 0,205—0,1575 = **0,0475%**.

Experiment 12 (Sept. 20).

The perfusion liquid contains a mixture of glucose and laevulose.
Reduction of solution passed 0,205%.
Reduction 0,1 c.c. urine of right kidney 0,115%.
Reduction 0,1 c.c. urine of left kidney 0,12%.
Retained by right kidney: 0,205—0,115 = **0,09%**.
Retained by left kidney: 0,205—0,12 = **0,085%**.

Experiment 13 (Sept. 21).

The perfusion liquid contains 0,07% glucose and 0,05% laevulose.
Reduction 0,1 c.c. of this solution 0,175% (average of 3 homonymous experiments).
0,1 c.c. urine of right kidney: 0,63 c.c. Iodine solution; reduction 0,1275%.
0,1 c.c. urine of right kidney: 0,62 c.c. Iodine solution; reduction 0,125%.
0,1 c.c. urine of left kidney: 0.65 c.c. Iodine solution; reduction 0,1325%.
Retained by right kidney: 0,175—0,1275 = **0.0475%**.
Retained by right kidney: 0,175—0,125 = **0,05%**.
Retained by left kidney: **0,175**—0,1325 = **0,0425%**.

Experiment 14 (Sept. 21).

The same perfusion liquid as in experiment 13.
0,1 c.c. urine of right kidney: **0,64** c.c. Iodine solution; reduction 0,13%.
0,1 c.c. urine of left kidney: 0,64 c.c. Iodine solution; reduction 0,1275%.
Retained by right kidney: 0,175—0,13 = **0,045%**.
Retained by left kidney: 0,175—0,1275 = **0,0475%**.

It is apparent from these experiments that the glomerulus epithelium which, as we have seen, is completely permeable to laevulose, has held back a quantity of glucose, which was also retained when the solution perfused contained glucose alone. The laevulose while passing itself does not or hardly influence the retention of glucose. With a little exaggeration one might this say that *the kidney separates the glucose from the laevulose by means of filtration.*

C. Galactose.

Experiment 15 (July 18).

The perfusion liquid contains 0,09% galactose.
Reduction (0,1 c.c.) of solution passed 0,07%.
0,1 c.c. urine of right kidney: 0,35 c.c. Iodine solution; reduction 0,055%.
0,1 c.c. urine of left kidney: 0,33 c.c. Iodine solution; reduction 0,052%.
Retained by right kidney: 0,07 – 0,055 = **0,015%**.
Retained by left kidney: 0,07— 0,052 = **0,018%**.

Experiment 16 (Aug. 22).

The perfusion liquid contains 0,1% galactose.
Reduction 0.1 c.c. of solution passed 0,07%.
0,1 c c. urine of right kidney: 0,3 c.c. Iodine solution: reduction 0,045%.
0,1 c.c. urine of left kidney: 0,28 c.c. Iodine solution; reduction 0,04%.
Retained by right kidney: 0,07—0,045 = **0,025%**.
Retained by left kidney: 0,07 – 0,04 = **0,03%**.

Experiment 17 (Aug. 23).

Reduction 0,1 c.c. of perfusion liquid 0,055%.
0,1 c.c. urine of right kidney: 0 25 c.c. Iodine solution; reduction 0,0325%.
0,1 c.c. urine of left kidney: 0,15 c.c. Iodine solution; reduction 0,0325%.
Retained by each kidney 0,055—0,0325 = **0,0225%**.

Experiment 18 (Aug. 23).

The same perfusion liquid as in experiment 17.
0,1 c.c. urine of right kidney: 0,25 c.c. Iodine solution; reduction 0,0325%.
0,1 c.c. urine of left kidney: 0,25 c.c. Iodine solution; reduction 0,0325%.
Retained by each kidney: 0,055 – 0,0325 = **0,0225%**.

All these experiments show that the kidney retains a slight quantity of galactose.

We shall now record a few experiments with a perfusion liquid the reduction of which approximately agrees with that of 0.1% glucose.

Experiment 19 (Aug. 23).

The perfusion liquid contains 0,15% galactose.
Reduction 0,1 c.c. of perfusion liquid 0,0975%.
0,1 c.c. urine of right kidney: 0.4 c.c. Iodine solution; reduction 0,07%.
0,1 c c. urine of left kidney: 0,4 c.c. Iodine solution; reduction 0,07%.
Retained by each kidney: 0,0975—0,07 = **0,0275%**.

Experiment 20 (Aug. 23).

The same perfusion liquid as in experiment 19.
0,1 c.c. urine of right kidney: 0,33 c.c. Iodine solution; reduction 0,0525%.
0,1 c.c. urine of left kidney: 0,32 c.c. Iodine solution; reduction 0,05%.
Retained by right kidney: 0,0975—0,0525 = **0,045%**.
Retained by left kidney: 0,0975—0,05 = **0,0475%**.

Here again it becomes clear that some galactose is retained. In the last experiment (20) the quantity is even comparatively large.

That galactose is retained is efficient in the same way as is the case with glucose. While glucose is a source of energy for muscular contraction; *galactose helps in the formation of the cerebrosides.*

As has been remarked earlier, frogs often present not inconsiderable differences in their capacity for retaining glucose. The time of year also has some influence. Therefore experiments were again made with frogs that had lived under the same circumstances as those of the experiments described above.

Experiment 21 (Aug. 24).
The perfusion liquid contains 0,1% *glucose.*
Reduction 0,1 c.c. of perfusion liquid 0,10%.
0,1 c.c. urine of right kidney: 0,24 c.c. Iodine solution; reduction 0,03%.
0,1 c.c. urine of left kidney: 0,22 c.c. Iodine solution; reduction 0,025%.
Retained by right kidney: 0.10—0,03 = **0,07%**.
Retained by left kidney: 0,10—0 025 = **0,075%**

Experiment 22 (Aug. 24).
The same perfusion liquid as in experiment 21.
0,1 c.c. urine of right kidney: 0,25 c.c. Iodine solution; reduction 0,0325%
0,1 c.c. urine of left kidney: 0,22 c.c. Iodine solution; reduction 0,025%.
Retained by right kidney: 0,10—0,0325 = **0,0675** %.
Retained by left kidney: 0,10—0,025 = **0,075** %.

It is clear that these frogs, which were placed in the same circumstances as those of experiments 16—20, retained *a much larger quantity of glucose than of galactose.*

It now seemed desirable to investigate whether the retention of galactose although this occurred in a much smaller degree than was the case with glucose, was governed by the same conditions as regards the composition of the Ringer-solution, as had formerly been found to apply to glucose. For this reason the following Ringer's solution was used: NaCl 0,7 %, **$NaHCO_3$ 0,02** %, KCl 0.01 %, $CaCl_2$ 0.0075 %. With the application of this solution glucose was at the time retained *to a maximum* of 0.03 %. Only when the quantity of $NaHCO_3$ was increased above 0.09 % on account of which the urine was no longer acid, there could be retained much more sugar; the urine could then even be free from sugar.

What would now be the result with galactose if the Ringer's solution also in this case contained 0,02 % *$NaHCO_3$ only?*

Experiment 23 (Sept. 14).
The perfusion liquid with only 0,02% $NaHCO_3$ contains 0,1% galactose.
Reduction of 0,1 c.c. of perfusion liquid 0,08%.
0,1 cc. urine of right kidney: 0,42 c.c. Iodine solution; reduction 0,075%.
0,1 c.c. urine of left kidney: 0,43 c.c. Iodine solution; reduction 0,0775%.
Retained by right kidney: 0,08—0,075 = **0,005%**.
Retained by left kidney: 0,08—0,0775 = **0,0025%**.

Experiment 24 (Sept. 14).
The same perfusion liquid as in experiment 23.
0,1 c.c. urine of right kidney: 0,43 c.c. Iodine solution; reduction 0,0775%.
0,1 c.c. urine of left kidney: 0,44 c.c. Iodine solution; reduction 0,08%.
Retained by right kidney: 0 08—0,0775 = **0,0025%**
Retained by left kidney: 0,08—0,08 = **0.**

We see thus that with galactose as well as with glucose, the quantity of $NaHCO_3$ in the perfusion liquid is of great importance for the permeability of the glomerulus epithelium. A solution containing a small quantity of $NaHCO_3$ causes galactose to pass in toto; if the perfusion liquid contains a physiological quantity of $NaHCO_3$ (0,285 %) then on an average 0.025 % galactose is retained. We say on an average, as frogs present individual differences.

Here follows a table which gives a summary of a series of other experiments with galactose in which however the quantity of $NaHCO_3$ was, accidentally, 0.2 % instead of 0.285 %. As had also formerly been found with glucose such a modification was of very little importance.

Retention capability of kidney for galactose.

Reduction perfusion liquid, containing ± 0.1 % galactose	Reduction urine of right kidney	Retained by right kidney	Reduction urine of left kidney	Retained by left kidney
0.0825 %	A 0.05 / 0.0475 } 0.049	**0.033** %	A 0.0525 / 0.0475 } 0.050	**0.0325** %
	B 0.0775 / 0.08 } 0.078	**0.038** „	B 0.0825 / 0.0825 } 0.0825	**0** „
	C 0.0675 / 0.065 } 0.0663	**0.0163** „	C 0.0675 / 0.0675 } 0.0675	**0.015** „
0.0725 „	D 0.05	**0.0225** „	D 0.0525	**0.02** „
	E 0.045	**0.0275** „	E 0.048	**0.0245** „
	F 0.047	**0.0250** „	F 0.045	**0.0275** „

When we make a study of this table it becomes clear;

1. that there is a difference in the power for retaining galactose in the various frogs A, B, C, D, E, and F.
2. that the individual differences range between 0 and 0,033%.
3. that the power of retention is more or less the same for the right and the left kidney of the same frog.

Here again as in the former experiments it is clear that galactose

does not, like laevulose, pass through the kidney altogether, but is generally retained to a slight degree.

This indicates in any case that galactose is used in the body. We know indeed that lactose is built up from dextrose and galactose and an article of ELSE HIRSCHBERG [1]) from WINTERSTEIN's laboratory has just appeared in which a certain affinity, that is wanting in laevulose, becomes clear between glucose and galactose in connection with the spinal cord.

We repeat, that the amount of galactose retained is in all the experiments *expressed in the percentage of glucose.* This is also the case with other sugars.

D. Mannose.

Experiment 25 (Aug. 20).

In the suitable perfusion liquid consisting of NaCl 0,5%, $NaHCO_3$ 0,285%, KCl 0,01% and $CaCl_2$ 0,02%, 0,1% mannose is dissolved.

0,1 c.c. perfusion liquid: 0,45 c.c. Iodine solution; reduction 0,0825%.
0,1 c.c. urine of right kidney: 0,48 c.c. Iodine solution; reduction 0,09%.
0,1 c.c. urine of left kidney: 0,46 c.c. Iodine solution; reduction 0,085%.
Retained by right kidney: 0,0825—0,09 = **0**.
Retained by left kidney: 0,0825—0,085 = **0**.

Experiment 26 (Aug 20).

The same perfusion liquid as in exp. 25.
0,1 c.c. urine of right kidney: 0,45 c.c. Iodine solution; reduction 0,0825%.
0,1 c.c. urine of left kidney: 0,44 c.c. Iodine solution; reduction 0,08%.
Retained by right kidney: 0,0825—0,0825 = **0**.
Retained by left kidney: 0,0825—0,08 = **0,0025%**.

Thus also in this experiment no mannose was retained.

Experiment 27 (Aug. 21).

0,1 c.c. perfusion liquid: 0,35 c.c. Iodine solution; reduction 0.0575%.
0,1 c.c. urine of right kidney: 0,37 c.c. Iodine solution; reduction 0.0625%.
0,1 c.c. urine of left kidney: 0.35 c.c. Iodine solution; reduction 0,0575%.
0,1 c.c. urine of right kidney: 0,35 c c. Iodine solution; reduction 0,0575%.

In this exp. also *no mannose is retained.*

Experiment 28 (Aug. 21).

The same perfusion liquid as in exp. 27.
0,1 c.c. urine of right kidney: 0,36 c.c. Iodine solution; reduction 0,06%.
0,1 c.c. urine of left kidney: 0,36 c.c. Iodine solution; reduction 0,06%.
Retained by each kidney 0,0575—0,06 = **0**.

In this experiment the Kidneys have also passed all the mannose.

From these experiments we may draw the conclusion that the glomerulus epithelium is totally permeable to mannose.

We shall now report some experiments made with *mutually isomeric disaccharides*: saccharose, maltose and lactose.

[1]) ELSE HIRSCHBERG: Zeitschr. f. physiol. Chemie, **100**, (1918).

These experiments seemed of importance because, amongst other reasons, their molecules are larger than those of the monosaccharides already considered. If the permeability was in connection with this size it would probably appear that the glomerulus epithelium was impermeable to the dissacharides.

E. Saccharose.

To estimate the quantity of saccharose in the perfusion liquid and in the urine, 0.1 cc. of the liquid was heated to 37° C. during $1^1/_2$ hours with 0.15 cc. hydrochloric acid 1 : 1 and the reduction of the solution thus obtained determined.

Experiment 29 (July 17).
The perfusion liquid contains 0,1% cane sugar.
After inversion it causes a reduction of 0,1275%.
0,1 c.c. urine of right kidney after inversion causes 0,13% reduction.
0,1 c.c. urine of left kidney after inversion causes 0,1325% reduction.
Retained by right kidney; 0,1275—0 13 = **0**.
Retained by left kidney: 0.1275—0,1325 = **0**.

Result: all the cane sugar has passed through.

Experiment 30 (July 17).
The same perfusion liquid as in exp. 29.
Reduction 0,1 c.c. urine of right kidney after inversion 0,1125%.
Reduction 0,1 c.c urine of left kidney after inversion 0,095%.
Retained by right kidney: 0,2275—0,1125 = **0,015**%.
Retained by left kidney: 0,1275—0,095 = **0,032**%.

Here follows a table in which several experiments are summarised.

There can be no doubt that the glomerulus epithelium has retained saccharose, either as such or in the form of glucose which has then been formed through the splitting of saccharose in the glomerulus epithelium, while the laevulose has passed into the urine. This latter alternative is of no great probability.

F. Maltose.

Experiment 35 (July 16).
In the suitable perfusion liquid with 0,285% $NaHCO_3$, 0,15% maltose is dissolved.
0,1 c.c. perfusion liquid: reduction 0,0825%.
0,1 c.c. urine of right kidney: reduction 0,07%.
0,1 c.c. urine of left kidney: reduction 0,07%.
Retained by each kidney: 0,0825—0,07 = **0,0125** %, thus hardly any at all.

Experiment 36 (July 16).
The same perfusion liquid as in exp. 35.
0,1 c.c. urine of right kidney: reduction 0,0875%.
0,1 c.c. urine of left kidney: reduction 0,0625%.
Retained by right kidney: 0,0825—0,0675 = **0,015**%.
Retained by left kidney: 0,0825—0,0625 = **0,02**%.

Retention capability of kidney for saccharose.
Experiments of September 17—23 1918.

Reduction perfusion liquid containing ± 0.1 % saccharose		Reduction urine of right kidney		Retained by right kidney	Reduction urine of left kidney		Retained by left kidney
Previous to inversion	after inversion	Previous to inversion	after inversion		Previous to inversion	after inversion	
Frog A 0.01 %	0.13 %	0.0325 %	0.09 %	**0.043 %**	0.025 %	0.0825 %	**0.0475 %**
„ B „			0.08	**0.05**		0.0775	**0.0525**
Frog A 0.015	0.125	0.02	0.0925	**0.0325**	0.02	0.0925	**0.0325**
„ B „		0.04	0.0925	**0.0325**	0.03	0.0925	**0.0325**
Frog A 0.025	0.14	0.03	0.1025	**0.0325**	0.03	0.10	**0.04**
„ B „	„		0.0875	**0.0525**		0.0825	**0.0575**
„ C „	„		0.09	**0.05**		—	—

Experiment 37 (Sept. 26).
The perfusion liquid contains 0,15% maltose.
0,1 c c. perfusion liquid: reduction 0,095%.
0,1 c c. urine of right kidney: reduction 0,105%.
0,1 c c. urine of left kidney: reduction 0,11%.
Retained by each kidney: *nothing*.

Experiment 38 (Sept. 26).
The same perfusion liquid as in exp. 37.
0,1 c.c. urine of right kidney: reduction 0,1%.
0,1 c.c. urine of left kidney: reduction 0,105%
Retained by each kidney: *nothing*.

Experiment 39 (Sept. 26).
The same perfusion liquid as in exps. 37 and 38.
0,1 c.c. urine of right kidney: reduction 0,0625%.
0,1 c.c. urine of left kidney: reduction 0,0625%.
Retained by each kidney: 0,095—0,0625 = **0,0325%**.

Experiment 40 (Sept. 27).
0,1 c.c. perfusion liquid: reduction 0,08%.
0,1 c.c. urine of right kidney: reduction 0,065%.
0,1 c.c. urine of left kidney: reduction 0,0625%.
Retained by right kidney: 0,08—0,065 = **0,015%**.
Retained by left kidney: 0,08—0,0625 = **0,0157%**.

It is clear from all these experiments that the quantity of maltose retained is in any case extremely small, notwithstanding the fact that

this dissaccharide is built up from two molecules of glucose, and that the glomerulus epithelium is permeable to glucose to a very slight degree.

G. Lactose and a mixture of glucose and lactose.

Experiment 41 (July 3).
0,2% Lactose is dissolved in the perfusion liquid.
0,1 c.c. of perfusion liquid: reduction 0,14%.
0,1 c.c. urine of right kidney: reduction 0,1375%.
0,1 c.c. urine of left kidney: reduction 0,14%.
Retained by right kidney: 0,14—0,1375 = **0,0025%**.
Retained by left kidney: 0,14—0,14 = **0**.

Experiment 42 (July 3).
0,1 c.c. perfusion liquid: reduction 0,11%.
0,1 c.c. urine of right kidney: reduction 0,11%.
0,1 c.c. urine of left kidney: reduction 0,1075%.
Retained by right kidney: 0,11—0.11 = **0**.
Retained by left kidney: 0,11—0,1075 = **0,0025%**.

Here again the kidney has retained no, or very little, lactose.

Experiment 43 (July 3).
The same perfusion liquid as in exp. 42.
0,1 c.c. urine of right kidney: reduction 0,1075%.
0,1 c.c. urine of left kidney: reduction 0,1075%.
Retained by each kidney: 0,11—0,1075 = **0,0025%**.

Here again neither of the kidneys retained lactose.

Experiment 44 (Juli 4).
0.1 c.c. perfusion liquid: reduction 0,14%.
0,1 c.c. urine of right kidney: reduction 0.1450%.
0,1 c.c. urine of left kidney: reduction 0,1325%.
Retained by right kidney: 0,14—0,1450 = **0**.
Retained by left kidney: 0,14—0,1325 = **0,0075%**.

Lactose has thus been retained by neither of the kidneys.

From this we may draw the conclusion that the glomerulus epithelium allows the lactose to pass altogether, although it is built up from glucose and galactose, of which compounds the former is retained to a large degree, and galactose as well, although less.

From a theoretical as well as from a clinical point of view it seemed of importance to investigate how the kidney would behave towards a mixture of lactose and glucose.

From these experiments we learn that the retention of glucose by the kidney is not influenced by lactose. The latter is passed while glucose is retained to the same degree as when there was no lactose present.

This result is in accordance with the use which for a

Retention capability of kidney for a mixture of 0.1 % glucose and 0.1 % sacch. lactis (lactose).
Experiments of September 24—26.

Reduction perfusion liquid	Reduction urine of right kidney	Retained by right kidney	Reduction urine of left kidney	Retained by left kidney
Frog A 0.1575 / 0.16	0.0825 / 0.0825	0.0765 %	0.0825 / 0.0825	0.076 %
Frog B „	0.09 / 0.095 } 0.0925	0.065 „	0.095 / 0.095 } 0.095	0.065 „
Frog C „	0.09	0.0675 „	0.0925	0.065 „
Frog D „	0.1	0.06 „	—	—
Frog A 0.16 %	0.096	0.064 „	0.104	0.056 „
Frog B „	0.10	0.06 „	0.10	0.056 „
Frog C „	0.104	0.056 „	0.094	0.066 „
Frog D „	0.096	0.064 „	0.10	0.060 „

considerable time has been made of lactose for clinical use to estimate the validity of kidneys and which is founded on the consideration that the healthy kidney easily passes lactose.

We thank Mr. R. Roelink for his able assistance during this research.

Summary and Conclusion.

1. The fact, now again affirmed, that, when we pass a Ringer's solution to which glucose has been added, through the kidney, this is retained by the glomerulus membrane, while salts, which are also crystalloids, are allowed to pass, has raised the question to what this contrast must be attributed.

2. In the first place we can think of the circumstance that glucose possesses a so much larger molecule, which could then impede the passage. If this hypothesis be correct then the disaccharides such as saccharose, maltose and lactose, which possess a still larger molecule ($C_{12}H_{22}O_{11}$) than glucose ($C_6H_{12}O_6$), would certainly not pass through. Experiments have however proved that the glomerulus epithelium is *permeable* to a large degree to these sugars, *even to raffinose* ($C_{18}H_{32}O_{16}$).

The permeability to lactose is perfect.

3. *Where the cause for the retention of glucose cannot be ascribed to the size of its molecule, we are bound to consider its structure or configuration.* There is the more occasion for doing this because its isomerics, laevulose and mannose, are allowed to pass altogether and galactose to a large degree, as has been proved by the experiments under discussion.

4. Glucose therfore occupies a unique position among the monosaccharides in regard to the glomerulus membrane. In other words *the glomerulus membrane can distinguish glucose from the other monosaccharides in a manner that reminds of the relation of sugars and ferments,* in connection with which EMIL FISCHER used the well-known simile of a *lock and key.*

In any case these experiments are again *a new illustration of the doctrine of stereoisometrics,* but now not as has thus far been the case, through facts of chemical but of *physiological nature,* belonging to permeability.

5. Not without theoretical and clinical importance seems the fact that the capability for retention of glucose is not modified when glucose and laevulose are simultaneously present in the perfusion liquid. The two sugars are simply separated as by a filter: the glucose remains behind, the laevulose is passed. This also appears to be the case with a mixture of lactose and glucose: the lactose passes completely into the urine and the glucose is retained by the glomerulus epithelium to the same degree as when there was no lactose present.

Groningen, September 1918. *Physiological Laboratory.*

Chemistry. — *"On the Significance of the Volta-Effect in Measurements of Electromotive Equilibria"*. By Prof. A. SMITS and J. M. BIJVOET. (Communicated by Prof. P. ZEEMAN).

(Communicated in the meeting of Sept 28, 1918).

Introduction.

Many physicists are of opinion that the Volta effect amounts to only a few milli volts, and that the electromotive force of an electrical cell resides practically exclusively in the potential differences metal-electrolyte, so that these alone need be taken into consideration; there are even those who think that the Volta-effect is theoretically zero. It is chiefly the German school that assumes that the Volta-effect may be neglected with respect to the potential difference metal-electrolyte. On the other hand a great number of investigators think they have found that the Volta-effect can constitute $\frac{1}{2}$, $\frac{3}{4}$ and even a still larger fraction of the electromotive force of a cell, and can accordingly amount to as much as 1 Volt.

The opinions concerning the value of the Volta-effect are therefore greatly divided, which is owing to the great difficulties which attend the determination of the Volta-effect.

In the application of the recent views about the electromotive equilibria [1]) to the Volta-effect it appears, that even though this effect should be small for metals in the state of internal equilibrium, it must become great for phenomena of polarisation, so that we may certainly not neglect the Volta-effect for these cases.

2. *The variation of the Volta-effect on polarization and passivation.*

The following equation holds for the potential difference of a metal in active state with respect to an electrolyte:

$$\Delta_{M\,active-L} = -\frac{0{,}058}{\nu} log \frac{K'_{M^{\nu\cdot}}(M_S^{\nu\cdot})_{active}}{M_L^{\nu\cdot}}$$

and thus we get for the passive state:

$$\Delta_{M\,passive-L} = -\frac{0{,}058}{\nu} log \frac{K'_{M^{\nu\cdot}}(M_S^{\nu\cdot})_{passive}}{(M_L^{\nu\cdot})}$$

from which follows.

[1]) Zeitschr, f. physik. Chemie **88**, 743 (1914).
" " " " **90**, 723 (1915).

$$\Delta_{M\,active-L} - \Delta_{M\,passive-L} = -\frac{0{,}058}{v} \log \frac{(M_S^{v^+})_{active}}{(M_S^{v^+})_{passive}}$$

For the Volta-effect holds the relation:

$$\Delta_{M_1-M_2} = \frac{\theta_{M_1} - \theta_{M_2}}{F}$$

from which follows in the same way as we have derived this for the potential difference metal-liquid, that the following equation holds for the Volta-effect:

$$\Delta_{M_1-M_2} = 0{,}058 \log \frac{K'_\theta(\theta_{M_1})}{(\theta_{M_2})}$$

so that:

$$\Delta_{M_1 active-M_2} - \Delta_{M_1 passive-M_2} = 0{,}058 \log \frac{(\theta_{M_1})_{active}}{(\theta_{M_1})_{passive}}$$

When we call:

$$\frac{(M_{1S}^{v^+})_{active}}{(M_{1S}^{v^+})_{passive}} = n$$

then also

$$\frac{(\theta_{M_1})_{active}}{(\theta_{M_1})_{passive}} = n$$

It follows from this that when on polarisation or passivation the change of the potential difference metal-electrolyte is:

$$\frac{0{,}058}{v} \log n$$

that of the Volta-effect amounts to:

$$0{,}058 \log n$$

hence v times the value.

What we measure is the sum of these two changes:

$$\frac{v+1}{v} 0{,}058 \log n.$$

Hence the part $\frac{v}{v+1}$ of this total change is due to the Volta-effect. This is, therefore, $\frac{1}{2}$ for a uni-valent metal, and $\frac{2}{3}$ for a bi-valent one etc.

When it is now borne in mind that on passivation through anodic solution the potential difference metal-electrolyte, as it is found by measurement, can change by an amount of 1 or 2 Volt. (e. g. 1.7 Volt, is found for iron), it follows from this that according to these considerations also the Volta-effect is subjected to a great change [1]).

[1]) It must still be pointed out that the case iron is certainly more intricate than the case considered above, because iron contains ions of different valency.

This throws, indeed, a peculiar light on the Volta-effect, for it now appears clearly that when really the Volta-effect for metals in the state of internal equilibrium should be zero or very small, which comes to the same thing as equality or nearly equality of the molecular thermodynamic potential of the electrons in those metals, this would be a special characteristic property according to the theory of electrons for metals in the state of internal equilibrium.

But apart from the value of the Volta-effect, in the logical development of the given theory of the electromotive equilibrium, the Volta-effect cannot be neglected. [1])

[1]) In this we will also point out that when the new views about the rôle of the electrons and the electromotive equilibrium are consistently applied to an arbitrary electrical circuit, we arrive at the result that everywhere where a potential difference occurs, a reaction takes place on passage of the current, in which the change in free energy of the reaction proceeding at that place, determines the value of the potential difference. It is known that according to the given theory at the places of contact metal-electrolyte this reaction consists: 1. of the splitting up of metal ions into ions and electrons; resp. of the formation of metal atoms from the electrically charged dissociation products mentioned here and 2 of the transition of ions and electrons from the metal phase into the electrolyte or vice versa. In this transition from one phase into the other the ions take a preponderant part.

When we consider the contact metal-metal, the just-mentioned transition consists to by far the greater part of the displacement of electrons from the metal M_1 to the metal M_2.

This view differs accordingly from that which is particularly met with in German handbooks, in which the opinion is embraced that the reactions during passage of the current exclusively take place at the places of contact metal-electrolyte.

It is clear that on application of the relation:

$$E = A - T\frac{dA}{dT}$$

to the transition of electricity from one metal into the other we may only conclude to:

$$A = T\frac{dA}{dT}$$

when in the process mentioned here no change of the thermo-dynamic energy takes place ($E = 0$).

LEBLANC does so in his handbook p. 227 (1914), and thus comes to the conclusion that the Volta-effect must be small, because $\frac{dA}{dT}$ is small.

According to the theory of electrons the difference in solubility heat of the electrons in the two metals will, however, have to be taken into account. In the isotherm transition of electricity between the two metals *the free energy* of the electrons will change, and be converted into electrical energy in case of a reversible process, the occurring change of the bound energy becoming manifest through the latent heat, which heat is the heat of PELTIER.

It is this quantity, the heat of PELTIER which, is represented by $T\frac{dA}{dT}$, and not the Volta-effect.

3. *The experimental electrical potential.*

When we measure the potential difference metal-electrolyte, we do so by the aid of an auxiliary electrode, e.g. a calomel or hydrogen electrode, in other words, we then make an electrical circuit, which is closed during the measurement. The electro-motive force of this circuit, in which we suppose the diffusion potential between L_1 and L_2 annulled, then becomes:

$$E = \Delta_{M_1-L_1} - \Delta_{M_2-L_2} - \Delta_{M_1-M_2}$$

in which $\Delta_{M_1-M_2}$ is a Volta-effect.

When in this we put $\Delta_{M_2-L_2} =$ zero (N. hydrogen-electrode), we get:

$$E = \Delta_{M_1-L_1} - \Delta_{M_1-M_2},$$

in which we shall call E the *experimental electrical potential.*

This expression, therefore, always contains the Volta-effect, and until this quantity is known, and until we introduce further suppositions about its value, we can of course not determine the difference $\Delta_{M_1-L_1}$ alone in this way, and consequently not the saturation concentration of the metal-ions either (which quantity is equivalent with NERNST's "Lösungstension"), which was calculated from:

$$E = -\frac{RT}{\nu_1 F} ln \frac{K_{M^{\nu}}}{(M_L^{\nu})} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which $\Delta_{M_1-M_2}$ has been neglected.

Nor can we draw conclusions about the order of the saturation concentrations of the metal-ions from the so-called tension series, until the Volta-effect shall be known. From the fact that the metal M_1 immersed in a normal solution of one of its salts appears on measurement to be more negative than the metal M_2, immersed in a 1 norm.-solution of a corresponding salt, we conclude namely that:

$$\Delta_{M_1-L_1} \text{ is more negative than } \Delta_{M_2-L_2},$$

but strictly speaking this conclusion is not permissible, because the measurement only says that $\Delta_{M_1-L_1} - \Delta_{M_1-M_2}$ is more strongly negative than $\Delta_{M_2-L_2}$.

We demonstrated that the equations for the exp. elec. potential in the form in which they contain the saturation-concentration of the ions or the "Lösungstension", have the drawback that the unknown Volta-effect occurs in them. It is entirely different with the new already before given equation, in which the *solubility product* of the metal or the *solubility quotient* of the metalloid occurs, and the same thing may be said of the electron-equation. [1])

[1]) Zeitschr. f. physik. Chemie **92**, 1 (1916).

Thus the following equation was namely derived:

$$\Delta_{M_1-L_1}-\Delta_{M_2-L_2}=\frac{RT}{\nu_1 F}ln\frac{(M_1^{\nu}{}_L)}{L_{M_1}}-\frac{RT}{\nu_2 F}ln\frac{(M_2^{\nu}{}_L)}{L_{M_2}}+\frac{\mu_{\theta_{M_1}}-\mu_{\theta_{M_2}}}{F} \quad (2)$$

we measure, however,

$$E=\Delta_{M_1-L_2}-\Delta_{M_2-L_2}-\Delta_{M_1-M_2} \quad (3)$$

and the Volta-effect being:

$$\Delta_{M_1-M_2}=\frac{\mu_{\theta_{M_1}}-\mu_{\theta_{M_2}}}{F}$$

it follows from this that:

$$E=\frac{RT}{\nu_1 F}ln\frac{(M_1^{\nu}{}_L)}{L_{M_1}}-\frac{RT}{\nu_2 F}ln\frac{(M_2^{\nu}{}_L)}{L_{M_2}} \quad (4)$$

in which the Volta-effect has been eliminated.

This equation enables us, therefore, to find the ratio of the solubility-products from the electromotive forces.

In the practical application the hydrogen electrode may be taken for the metal M_2, and for L_{H_2} the value may be substituted which had already been given before, viz. $10^{2\times-48}$ [1]);
in this case we get:

$$E=\frac{0{,}058}{\nu}log\frac{(M_L^{\nu})}{L_M}-2{,}8 \quad (5)$$

When equation (4) is compared with the expression for E, which contains the saturation-concentrations of the metal-ions found by substituting the values of $\Delta_{M_1-L_1}$ and $\Delta_{M_2-L_2}$ given by equation (1) in equation (3), which gives:

$$E=-\frac{RT}{\nu_1 F}ln\frac{K_{M_1}{}^{\nu}}{(M_1{}_L^{\nu})}+\frac{RT}{\nu_2 F}ln\frac{K_{M_2}{}^{\nu}}{(M_2{}_L^{\nu})}-\Delta_{M_1-M_2} \quad (6)$$

the great advantage which equation (4) resp. (5) have over equation (6) is very apparent, for the latter equation contains a still unknown Volta-effect. While the construction of a series of the potential differences metal-electrolyte is not yet possible, on account of our ignorance of the Volta-effect, equation (5) enables us to draw up a series for the solubility-products of the metals, and from such a relation there may be found a series for the solubility-quotients of the metalloids, as has been done already [2]).

The determination and order of this series is of course the same as that of the socalled tension-series, which gives the order in which

[1]) Zeitschr. f. physik. Chemie **92**, 1 (1916).

[2]) Smits and Lobry de Bruyn. Verslag Kon. Ak. **26**, 270.

the metals oust each other from equally concentrated solution. We demonstrated, however, that in a theory which assumes the existence of Volta-effects, the quantities derived from the electromotive equilibria should not be considered as the saturation-concentrations of the metal-ions or as the "Lösungstensionen".

The electron equation gives a relation that agrees with equation (2), viz.:

$$\Delta_{M_1-L_1} - \Delta_{M_2-L_2} = -\frac{RT}{F} ln \frac{(\theta_{L_1})}{(\theta_{L_2})} + \frac{\mu_{\theta_{M_1}} - \mu_{\theta_{M_2}}}{F},$$

so that

$$E = -\frac{RT}{F} ln \frac{(\theta_{L_1})}{(\theta_{L_2})}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Of course the Volta-effect does not occur in this relation either, so that when a definite value is assumed for the electron conc. (θ_{L_2}) in the electrolyte in which the hydrogen electrode is found, the electron-concentration (θ_{L_1}), which belongs to the other electromotive equilibrium, can be found.

Equations (4) and (7) are, therefore, not approximate, but perfectly rigorous, which also appears from the consideration that equation (7), from which (4) is found by substitution of $\frac{L_M}{(M_L^{\nu})}$ for $(\theta_L)^{\nu}$, follows immediately from the condition for electron equilibrium between the liquids E_1 and L_2. The derivation of this electromotive force as sum of the potential differences occurring in the circuit was accordingly only followed here in order to keep in agreement with the derivation given before.

4. As appears from a previous communication [1]), the experimental elec. potential of a Ni-electrode, immersed in an electrolyte through which hydrogen was led, was found equal to the hydrogen potential. In order to account for this fact that the electromotive force E of the closed circuit nickel-electrode-electrolyte-hydrogen-electrode is zero according to our considerations we must show, as follows directly from equation (7), in what way the electron concentration of the nickel equilibrium has become equal to that of the hydrogen equilibrium.

This phenomenon is already explained in a very simple way [2]) by the assumption that the nickel-electrode is, at least superficially,

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[1]) Smits and Lobry de Bruijn loc. cit.

[2]) Smits and Lobry de Bruyn loc. cit.

disturbed, which disturbance for a very inert metal will go so far till the electron concentration of the nickel equilibrium in the electrolyte has conformed to the electron concentration of the hydrogen equilibrium, and has become almost equal to it.

For the limiting case of an ideal inert metal this equality will become perfect, as is also required by E becoming zero in this case.

In a former discussion of these phenomena the neglect of the Volta-effect led on the other hand to the conclusion that the electromotive force of the circuit, in case of equal electron-concentrations in the electrolyte of the two metal equilibria, would be zero *only in approximation.*

It will have become clear through what precedes that while up to now the Volta-effect has been neglected in the Δ,X-figure, it is better to draw the experimental potential E (e.g. with respect to hydrogen) as ordinate instead of the potential difference metal-liquid; then the thus obtained E,X-figures are perfectly rigorous. They then indicate that for the three-phase equilibrium the two electrodes possess the same experimental potential; the difference between their potential differences with the electrolyte then being equal to the Volta-effect between the two metals.

In this point the following equation then holds:

$$\Delta = \frac{0{,}058}{\nu_1} \log \frac{\left(M_{1L}^{\nu}\right)}{L_{M_1}} = \frac{0{,}058}{\nu_2} \log \frac{\left(M_{2L}^{\nu}\right)}{L_{M_2}}$$

or

$$\frac{L_{M_2}^{\nu}}{L_{M_1}^{\nu}} = \frac{\left(M_{2L}^{\nu}\right)^{\nu_1}}{\left(M_{1L}^{\nu}\right)^{\nu_2}}$$

so that from the ratio of the solubility-products the situation of the coexisting electrolyte immediately follows. Though this has been shown already before, this circumstance is once more pointed out here, to make clear that equation (6) is of no use to us here, for this equation gives for the three-phase equilibrium:

$$-\frac{0{,}058}{\nu_1} \log \frac{K_{M_1}^{\nu}}{\left(M_{1L}^{\nu}\right)} = -\frac{0{,}058}{\nu_2} \log \frac{K_{M_2}^{\nu}}{\left(M_{2L}^{\nu}\right)} + \Delta_{M_1-M_2}$$

In this form the equation contains, however, the unknown Volta-effect and the unknown saturation-concentrations, so that a calculation as above is not possible.

Summary.

By application of the more recent views on the electromotive

equilibria to the Volta-effect the result was obtained that on polarisation and passivation the *change* in the Volta-effect must be great according to this theory, and that in these phenomena the Volta-effect would even constitute the greatest part of the total change in the electromotive force. It was further pointed out that if the Volta-effects between metals in which the state of internal equilibrium prevails, should be really very small, on which the opinions are still divided, but which has, indeed, become probable by the experiments made of late, this would have to be considered according to the theory as a very characteristic property for metals in the state of internal equilibrium.

As in principle the Volta-effect at any rate in the given theory of the electromotive equilibria may not be neglected, it was here taken into account. The quantities that are found on measurement of potential differences metal-electrolyte, and which are here called *experimental electrical potentials*, always contain an unknown Volta-effect, which is the cause that from the said potentials the saturation concentrations of the metal ions (resp. the "Lösungstensionen") cannot be calculated.

The newly derived relations, in which the *solubility product* of the metal, the *solubility quotient* of the non-metal resp. the electron concentration in the electrolyte occurs, do not contain the Volta-effect, however, and enable us to draw up a *solubility-product-series* of metals, and a *solubility-quotient-series* of metalloids, as has indeed already been done, from which conclusions can be drawn about the chemical and electrochemical behaviour of the elements.

Amsterdam,
Sept. 1918.

General and Anorg. Chem. Laboratory
of the University.

Zoology. — "*Androgenic origin of Horns and Antlers.*" By Prof. J. F. VAN BEMMELEN.

(Communicated in the meeting of September 29, 1918).

In his excellent work: "die Säugetiere", MAX WEBER gives as his opinion about the origin of the cephalic armament of numerous Ungulates, that horns and antlers originally started in both sexes as defensive weapons against enemies, but later on more and more came to be used as instruments of offence by the males in their fights for the females, and so either have grown an exclusive attribute of the male sex, or at least have developed much more strongly than in the female sex.

In this instance therefore WEBER evidently shares the opinion pronounced by TANDLER and GROSS in their paper: Die biologischen Grundlagen der sekundären Geschlechtscharaktere, where they say: "All secondary sexual features were originally specific features, properties therefore, characteristic of a certain species, even of a whole order of Vertebrates, without their primarily having any connection with the genital sphere."

In their commentary on this proposition they remark: "Hitherto in the morphology of the sexual characteristics too little attention has been paid to the question, how much of them is peculiar not for the sex, but for the species." As a special example they cite the case of the horns of Cavicornia, "which do not constitute a sexual characteristic in themselves, but only in their shape, which differs for males and females, whereas on the contrary it is identical in masculine and feminine castrates."

"The same is the case with the hairiness in man. We have been able to show that such an eminently secondary sexual feature of the male sex, as the beard, is also found in old castrates, but there in form and extension resembles that of old women."

According to TANDLER and GROSS the question should not be formulated: "Is an organ a secondary sexual feature", but: "How much in the development of an organ is specific, how much sexual."

Though this assertion might be granted, yet I believe to be justified in opposing to it another view: viz. that the evolution has been just the reverse; the cephalic armament arising in males as a means of attack in their duels for the females, and afterwards passing to

the latter in a more or less reduced form and only in some of the species (according to the well-known rules of monosexual transgressional heredity), the result being that the horns now could be used in both sexes as means of defence.

As arguments in favour of this opinion may be cited:

1. In Deer the antlers are absent in all females except those of the Reindeer, and precisely in this species a useful application of the horns, not connected with any sexual function, and alike for both male and female may be conceived, viz. the digging up of food from under the snow, though some authorities (e.g. Brehm) deny this function.

2. Among Antilopes, besides genera in which both sexes are horned, others occur in which only the male possesses these attributes, and in the majority of cases the horns of the females are smaller than those of the males. The latter moreover show a tendency to hypertrophic growth, just as is the case with deer, leading to unwieldy size or to sundry strange shapes (e.g. screwlike contortions) which seem to stand in direct contradiction with the requirements of practical use.

3. The same is the case with Cattle, Sheep and Goats, as shown by the four-horned goat, or the excessive development of the horns in the carbon and other buffaloes.

4. Even in Giraffes, whose minute pedicles with their small os cornu may probably be considered as rudiments of formerly better differentiated antlers (compare Ocapia and Sivatherium) the males possess higher and stronger hornstumps than the females, and moreover the unpaired nasal knob. In Ocapia on the other hand the horns are primarily absent in the female, and the same was probably the case during their whole life in those of Siva-, Hellado- and Samotherium.

5. The more original kinds of Ruminants: the Tragulidae and Camelidae, are destitute of horns, and so were the oldest and primitive extinct Artiodactyla (Pantolestidae, Anoplotheridae) as well as all Nonruminantia amongst them. The oldest fossil deer likewise did not yet possess antlers, as the muskdeer does up to this day, though the miocene Palaeomeryx according to Rütimeyer and Schlosser was already provided with them. Perhaps this might be considered as an indication that the tendency to the formation of antlers arose independently in the tribe of the Cervidae, but also as the manifestation of a far older hereditary inclination to the production of bony frontal appendages of the male. This tendency then must have been in abeyance in the tribe of the Artiodactyla in general; in Deer

it probably reappeared, and from thence continually increased in potency and complication.

This conception according to my view is strengthened by the fact, that also among the Suidae a tendency to the formation of bony protuberances on the dorsal side of the skull undoubtedly occurs and appears in stronger manifestation in the male sex than in the female, as is shown by the monstrous skull of the male African wart-hog.

6. In Protoceratinae likewise the skull of the male only was provided at its upper side with a complete set of paired bony excrescences.

7. According to MARSH both sexes of the Dinoceratidae possessed these bony protuberances (and the large dagger-like tusks besides), yet in the male they grew unto a larger size than in the females. We might conclude from this that the tendency to the production of bony knobs on the skull is even older than the separation of Ungulates into Artio- and Perissodactyles.

8. The annual shedding of the antlers and their regeneration in Cervinae is apparently connected with the rut. The same appears to be the case in Antilocapra.

9. The abovementioned bony processes on the head of Giraffidae, (sensu latiori) Suidae, Protoceratinae and Dinoceratidae, cannot reasonably be considered as really practical weapons, as they are far too cumbrous and hypertrophic for that. Neither can this be the case with the antlers of most Deer or the horns of numerous Antilopes, Cattle, Sheep, and Goats.

On the other hand they wear to a very high degree the character of sexual attributes, in their exuberance, unpractical build, curious complication, obviousness and variability.

10. In the first (primitive) members of the Ruminantia antlers and horns apparently arose at a relatively late stage, though this may be further removed in the geological past than is generally supposed. In any case the appearance or return of this feature is younger than the remaining peculiarities of Artiodactyla.

As I consulted the literature on the subject, I found that in considering horns and antlers as sexual attributes, I had come to a similar conclusion as the well-known popular author on questions of evolution in Zoology, BÖLSCHE, has set forth in his Tierbuch IV, der Hirsch. Yet on the first cause of the origin of frontal appendages our opinions disagree, for BÖLSCHE sees in the excrescences on the roof of the skull of so many Ruminants nothing more than originally purely ornamental attributes, and ascribes their birth to a periodical exuberance of energy of growth, manifesting

itself in exostoses of the bones of the cranial roof, especially and at last exclusively of the frontal bones. In his opinion the periodicity of this surplus of growth-energy keeps time with the rut of the males, but its presence should be ascribed to the regression of another differential feature of the male sex, viz. the ensiform tusks, as they still occur in Suidae, Tragulidae and the unhorned muskdeer.

The strongest expression of this opinion is given by BÖLSCHE in the words (p. 88) "The pedicle is no weapon." On p. 89 he continues: "As we saw, the idea "weapon" cannot be applied without reserve to the beam, although it may occasionally be used as such. Far exceeding that application and evidently its real nature, this beam is an ornamental product, a somatical arabesk, abstract from all usefulness, rhythmic in structure, with an inherent connection with the erotic side of life. The pedicle, in principle a product of the skull like the beam, cannot by any means be considered as a weapon, at the same time however it does not want erotic connections."

Against these views I think objections may be raised in two respects. In the first place there is no plausible reason, why the origin of horns and antlers should not be connected with single combats between males belonging to the same species, in which the more primitive mode of fighting with tusks (as still found among hogs) was gradually replaced by knocking of the foreheads against each other.

The question, whether this new custom was the immediate or the indirect cause of the exostotic hypertrophic process (Lamarckism versus Darwinism) may be passed over in silence here as in all similar cases. Nor do I want to deny that exuberant growth, in cooperation with periodical sexual maturity, exerted an important influence on their development, as it still does every time the antlers are shed and renewed: we need only remember how profoundly this renewal is disturbed by every injury to the male sexual glands.

In the same way, BÖLSCHE's verdict: "The pedicle is no weapon", seems to me to be liable to serious doubt. Already in itself, the comparison of the long pedicle in the Muntjac-deer with the shorter ones of the remaining Cervidae leads to the conception, that the pedicle should be considered as an organ in a state of regression. To the same consideration leads a survey of the extinct Deer: in the middle-miocene Palaeomeryx-species no separation exists between pedicle and beam, they only show a long bony outgrowth of their frontal bones, slightly forked at its top. This excrescence therefore might be considered as a pedicle of extraordinary length. In the

first fossil deer showing a separate pedicle, the latter is very long, like that of the existing Muntjac.

In accordance with this observation is the fact that in Sivatherinae no rose can be detected near the base of their gigantic and ramified antlers, which therefore as a whole might be considered as pedicles.

Starting from the fact that in recent Giraffes the small hornstumps permanently retain their covering of hairy skin, the same may have occurred in their extinct allies: the Sivatherinae, and perhaps likewise in the first antlered ancestors of Deer. The separation of the latter's antlers into pedicle and beam, combined with the phenomena of yearly shedding and regeneration, the "rubbing" of the "velvet", in short the entire process of the renovation of the antlers, so inconvenient and dangerous for the stag, might then have developed from a similar primitive condition as that in Sivatherinae, where the frontal bony outgrowths, clad with hairy skin, must gradually have increased in size and complication. If one applies to the latter the designation "pedicle", it follows that for them just the contrary might be true of what is implied in BÖLSCHE's assertion: viz. the pedicle would have originated as a weapon and only lost this function in the Giraffes proper.

In the second place I cannot see sufficient reason for accepting such an intimate and strict connection between the regression of the tusks in the upper jaw of the stag and the progression of their antlers, as necessarily follows from the supposition that a surplus of growing energy should pass from those tusks to the frontal bones.

Against this hypothesis it may be objected that the male Protoceratinae as well as the Dinoceratidae were provided with powerful tusks, largely protruding from their mouths, and yet had a whole range of paired and single bony knobs and projections on the roof of their skulls, somewhat like those still found in the (male) warthogs. Among deer the male Muntjac still possesses strong tusks projecting downward and outward out of the mouth from under its upper lip, and yet carries well-developed, though simple antlers.

There is moreover little reason for the assumption that in Cavicornia the same course of events should have taken place as in Cervicornia, viz. a regression of large tusks, going hand in hand with an increased growth and a higher complication of frontal appendages, and yet the origin of these excrescences may be attributed to similar causes in all Horn- and Antlerbearing Ungulates.

On the other hand there is nothing incomprehensible in the fact that the upper tusks of Deer should have been reduced, as soon as they were no longer used as weapons, because the male Deer got accus-

tomed to a new mode of fighting which made them acquire antlers.

This vicariating development can be understood, without taking refuge to such an intricate correlation between upper tusks and antlers, as BÖLSCHE does, where he speaks of a surplus of energy of growth, set free by retardation in the development of tusks, and manifesting itself in hypertrophic excrescences on the frontal bones.

When we proceed in this course of thought, the question unvoluntarily arises, if the far higher development of tusks in the male sex of so many species of mammals might not be considered as a support for the abovementioned hypothesis about the androgenic origin of frontal appendages. The difference between the two phenomena lies especially in the fact, that with canines modified to tusks or incisors prolonged into darts, only their stronger development and differentiation need be ascribed to influences of sexual, especially male nature, while for horns and antlers also their first appearance had to be traced to this same cause. But this does not exclude that the tusks of Elephants and Cetacea, the canines of so many Apes, Carnivores and Ungulates etc. find their most plausible explanation in the assumption, that in so far as they are larger than the other teeth and also differ in shape and position from the incisors and molars, they may be considered as an acquisition of the male sex, which afterwards passed to the female, but in reduced proportion, and so to a certain extent again lost its monosexual character. Especially the growth far over the limits of practical fitness may be adduced as an argument for this hypothesis; we should only remember the tusks of the Mammoth curled up in a complete circle, the gigantic canines of the male walrus or the usually unilateral dart of the male Narwhal.

In this connection I should like to move the question whether the uncouth tusks of the extinct Sabretoothed tiger (Machairodus), might not have formed a special attribute of the male sex, as beyond doubt they were far from practical in defence as well as offence. With certainty this is the case with the Walrus. Also the upper canines of the Babirussa, which perforate the upper lip, and are curled up dorsally and backwards, give us a good example of hypertrophic growth far beyond the limits of real usefulness.

Groningen, September 1918.

Chemistry. — "*On the estimation of the geraniol content of citronella oil*". By Dr. A. W. K. DE JONG. (Communicated by Prof. VAN ROMBURGH).

(Communicated in the meeting of September 29, 1918).

The chemists of the firm of SCHIMMEL & Co. have a method for the estimation of geraniol in citronella oil which in the "Bericht" of this firm of October 1899, 20 and also in that of October 1912, 39, is described as follows: "Etwa 2 g Phthalsäureanhydrid und 2 g des zu untersuchenden Oels werden mit 2 ccm Benzol zwei Stunden in einem Kolben, wie er zu Acetylierungen benutzt wird, auf dem Wasserbad erwärmt, dann erkalten gelassen und mit 60 ccm wässriger Halbnormal-Kalilauge 10 Minuten geschüttelt. Der Kolben ist hierbei mit einem eingeschliffenen Glasstopfen verschlossen. Nach dieser Zeit ist alles Anhydrid in neutrales phthalsaures Kali und der saure Geraniolester in sein Kalisalz übergeführt worden. Nun wird das überschüssige Alkali mit Halbnormal-Schwefelsäure zurücktitriert. Zieht man dann von der Menge Alkali, die der eingewogenen Phthalsäure entspricht, die für den Versuch verbrauchte Menge ab, so erfährt man, wieviel Alkali dem an Phthalsäure gegangenen Geraniol äquivalent ist, woraus der Prozentgehalt an Geraniol zu berechnen ist".

This method is at the outset subject to suspicion, since it is based on the assumption that geraniol is quantitatively esterified by phthalic anhydride, whereas this is not even the case with acetic anhydride (98.5 p.c. was found to be esterified).

For the following experiments a very pure phthalic anhydride was prepared; 1 grm. was neutralised by 135 c.c. $^1/_{10}$ n KOH, whereas 135.1 c.c. was calculated for the pure substance.

The citronellal used was isolated from citronella oil by means of the bisulphite compound. The sp. gr. at 26° was 0.8526; $[\alpha]_D = +10°21'$. It was faintly acidic; 1 grm. was neutralized by 0.2. c.c. $^1/_{10}$ n KOH.

The geraniol was isolated from Palmarosa oil by means of the calcium chloride method. The sp. gr. at 26° was 0.8752.

To the mixture of anhydride, citronellal or geraniol 4 c.c. of benzene was added. After heating, the flask was cooled rapidly, so

that the crystals which separated remained small and dissolved readily in ½ n KOH. The phenolphtalein was added as powder.

Duration of heating: 2 hours.

Temperature of the water bath in degrees C.	Quantity of phthalic anhydride used	Quantity of citronellal used	cc. N/10 KOH		Difference
			found	calculated	
85	1.2600 grm.	0.4450 grm.	169.9	170.1	0.2
85	0.9480 „	0.5120 „	127.6	127.95	0.35

At this temperature citronellal is not attacked by the anhydride, or scarcely so.

Temperature of the water bath in degrees C.	Duration of heating	Quantity of phthalic anhydride grm.	Quantity of geraniol used grm.	Quantity of geraniol found	
				grm.	%
99	2 hours	2.1740	0.4645	0.4158	89.5
88	2 „	2.1810	0.4960	0.4575	92.2
84	2 „	2.1300	0.4860	0.4481	92.2
77	2 „	2.1160	0.4950	0.4527	91.5
88	2 „	1.0550	0.4490	0.4073	90.7
82	3 „	1.6930	0.4515	0.4173	91.4

This showed clearly that the esterification of the geraniol had not been complete. The mixture does not boil, however, and I thought that this might possibly be the cause of the shortage. Hence a few further estimations were made in which the heating was done with a small flame.

Duration of heating: 2 hours.

Quantity of phthalic anhydride used in grm.	Quantity of geraniol used in grm.	Quantity of geraniol found	
		grm.	%
2.0540	0.8890	0.8193	92.2
2.0495	0.4970	0.4497	90.5

Nor did an increase in the benzene from 4 c.c. to 8 c.c. change the result. Found: 90.6 %.

With this way of heating the citronellal is also attacked more extensively.

Quantity of phthalic anhydride used in grm.	Quantity of citronellal used in grm.	Number of c.c. N/10 KOH		
		found	calculated	difference
0.9490	0.5950	126.5	128.1	1.6
1.2850	0.5050	171.15	173.45	2.3

Evidently the scientific investigators of the firm of SCHIMMEL & Co only worked with mixtures of geraniol, citronellal and limonene. As the following estimations show, results are obtained with mixtures of geraniol and citronellal which differ but little from the true values.

Temperature of the water bath in degrees C.	Quantity of phthalic anhydride used in grm.	Quantity of geraniol used in grm.	Quantity of citronellal used in grm.	Quantity of geraniol found	
				grm.	%
88	2.4883	0.5610	1.3602	0.5480	97.7
82	2.1090	1.0115	0.8120	1.0255	101.4
86	2.3600	1.3745	0.7560	1.3883	101.0

That the presence of the citronellal should lead to better esterification of the geraniol is very improbable; since in the previous experiments about 92 % of the geraniol was esterified, the amount of citronellal esterified in the last three estimations would be 0.032 grm., 0.095 grm. and 0.123 grm. respectively.

It would appear from this that the amount of citronellal esterified increases with that of the geraniol. The cause for the esterification of citronellal in the presence of geraniol must be sought in the formation of the acid phthalic ester of geraniol. Phthalic acid itself has little effect because it is only slightly soluble in benzene. It is well known that citronellal is very sensitive to acids, being converted by them into isopulegol.

In acetylating citronellal without sodium acetate the same phenomenon is observable when working with mixtures of acetic acid and acetic anhydride.

An indirect method was employed corresponding to the way in which the geraniol was estimated by phthalic anhydride. The apparatus consisted of a small flask with long neck to which a ground in U-tube was attached. Into the flask there was always weighed 2 c.c. of citronellal and 2 c.c. of the acetic anhydride mixture. 5 c.c. of $^1/_2$ n KOH were placed in the U-tube, which was fitted with a soda-lime tube.

Acetic anhydride content of the mixture %	Quantity of citronellal esterified in %		Mean
	A	B	
95.0	28.3	32.0	30.2
75.9	51.9	55.8	53.9
53.6	70.0	68.8	69.4
31.2	59.4	—	59.4
15.25	43.7	41.8	42.5
Acetic acid of 97.2 %	30.9	30.9	30.9

When the duration of heating was increased from 2 hours to 3 hours, the amount of citronellal esterified was also increased. On using 95 % acetic anhydride, 40.1 % and with 52.9 % anhydride, 76.7 % of the citronellal was esterified.

Clearly the presence of acetic acid in the mixture favours the esterification of the citronellal. It might be concluded, that the citronellal which is not esterified, is nevertheless transformed in some other way, for instance into a terpene or similar body. In order to investigate this point larger quantities of the various products were prepared in the manner in which the estimation of the so called total geraniol content is carried out (Bericht of SCHIMMEL & Co April 1910, 155).

A portion of each product was examined by means of phthalic anhydride for the presence of alcohols, of another portion the saponification number was determined in the ordinary way and a third portion was acetylated by the indirect method, by heating 2 c.c. of the product with 2 c.c. of acetic anhydride of 95 % and 0.2 grm. of sodium acetate for 3 hours.

Acetic anhydride content of the mixture %	Number of c.c. of N/10 phthalic acid esterified	Saponification number	% acetylated according to indirect method
95.0	1.6	124.3[1])	67.7
53.6	—	219.0	13.4
31.2	1.0	180.3	19.8
15.25	—	129.0	28.4
Acetic acid of 97.2 %	1.3	98.0	38.6

Altogether the following amounts of citronellal were therefore acetylated.

95.0 %	37.7 + 74.6 = 112.3
53.6	72.0 + 16.0 = 88.0
31.2	57.3 + 22.9 = 80.2
15.25	39.2 + 31.6 = 70.8
Acetic acid of 97.2 %	29.1 + 41.6 = 70.7

The original use of 95 % anhydride therefore leads to partial formation of a diacetate, while the other acetic anhydride mixtures do not yield 100 %, which would indicate that these cause, in addition, the formation of hydrocarbons terpenes.

The presence of geraniol, like that of acetic acid, leads to a better esterification of the citronellal, as was the case in the estimations with phthalic anhydride.

While according to the indirect method with acetic anhydride of 95 % in two hours only 30.2 % of citronnellal was esterified, mixtures with geraniol gave the following result:

Quantity of geraniol grm.	Quantity of citronellal grm.	Quantity esterified grm.
0.4940	1.1365	0.9478
0.8275	0.9465	1.3195
1.3390	0.8260	1.8740

Assuming that in the mixture 98.5 % of the geraniol is esterified,

[1]) This higher figure is most probably due to increased absorption of water, when working on a large scale.

as was found in the experiment with the pure substance, we can calculate that the following quantities of citronellal were esterified: 40.6, 53.3 and 67.2.

It further follows that no complete esterification is possible without the use of sodium acetate. Even by increasing the duration of heating to 10 hours only 93.3 % of the geraniol-citronellal mixture (1 c.c. to 4 c.c.) was esterified.

Finally some estimations were made in which for every 2 c.c. of citronellal 0.2 grm. of sodium acetate (previously melted) was employed.

Acetic anhydride content of the mixture %	Citronellal esterified in %		Mean
	A	B	
95.0	95.0	95.3	95.1
88.3	93.5	—	93.5
75.9	90.0	92.4	91.2
52.9	84.4	—	84.4
31.2	54.5	—	54.5
15.25	40.2	—	40.2
Acetic acid of 97.2 %	30.3	31.4	30.8

Hence the presence of sodium acetate increased the quantity of acetylated citronellal in those mixtures which contained 53—95 % of anhydride, and did not affect the others. The rise of temperature due to the addition of the sodium acetate is not the cause of the improved acetylation in the former mixtures, since heating the mixture in a sealed tube without sodium acetate to the same temperature (about 149°) did not result in better esterification. The curve shows, however, that when sodium acetate is used, the presence of acetic acid is harmful, whereas, in the absence of sodium acetate, the acid has a favourable effect up to a certain concentration. It follows therefore that in using sodium acetate we induce a different reaction from that which occurs in the absence of this salt.

Buitenzorg, 27 May 1918.

Mathematics. — "*Observations on the expansion of a function in a series of factorials.*" II. By Dr. H. B. A. BOCKWINKEL. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of September 29, 1918).

5. We now consider another example of NIELSEN's theorem, not belonging to the cases mentioned under N°. 4 of the remarks made in the preceding paragraph. We choose

$$\varphi(t) = \frac{1}{e^{i\theta} - t}$$

where θ is a number between 0 and 2π, not equal to one of these numbers. For this function we have

$$\lambda = -\infty \quad , \quad \lambda' = 0,$$

the first of these equations resulting from the fact that $t = 1$ is an ordinary point of the function. It is further easily found that the nth derivative of $\varphi(t)$ satisfies the equation

$$\frac{\varphi^{(n)}(t)(1-t)^{\delta+n-1}}{\Gamma(\delta+n)} = \frac{n\Gamma(n)}{\Gamma(\delta+n)}\left(\frac{1-t}{e^{i\theta}-t}\right)^n \times \frac{(1-t)^{\delta-1}}{e^{i\theta}-t} \quad . \quad . \quad (21)$$

The modulus of the expression $\frac{1-t}{e^{i\theta}-t}$ is given by the relation

$$\left|\frac{1-t}{e^{i\theta}-t}\right| = 1 - \frac{2t(1-\cos\theta)}{1-t+\sqrt{1-2t\cos\theta+t^2}} \quad . \quad . \quad . \quad . \quad (22)$$

and it is not very difficult to see that it increases monotonously from the value 0 to 1, if t decreases from 1 to 0.

We divide the interval (0,1) of t into two parts, $(0, \nu)$ and $(\nu, 1)$, where ν is a number given by

$$\nu = n^{\delta_1 - 1} \qquad (0 < \delta_1 < 1) \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

so that ν depends on n and approaches to zero as a limit when n becomes indefinitely large. The positive number δ_1 is at our disposal and will be fixed immediately. The maximum value of the modulus (22) then differs from unity by a quantity greater than

$$kn^{\delta_1 - 1}$$

if t lies in the second interval, k being a certain positive number, which is independent of n and t; thus we have in this interval

$$\left|\frac{1-t}{e^{i\theta}-t}\right|^n < e^{-kn^{\delta_1}}$$

so that the left-hand member of (21) for these values of t approaches *uniformly* to zero for $n = \infty$ (the factor $n\,\Gamma(n) : \Gamma(\delta + n)$ is only equivalent to $n^{1-\delta}$ and does therefore not affect this statement). The integral

$$\int_\nu^1 \frac{n\Gamma(n)}{\Gamma(\delta+n)}\left(\frac{1-t}{e^{i\theta}-t}\right)^n \frac{(1-t)^{\delta-1}}{e^{i\theta}-t}\,dt$$

consequently has zero for its limit if n increases indefinitely, however small the value of δ may be fixed.

For the interval $(0, \nu)$ we have, independently of t and n,

$$\left|\frac{n\Gamma(n)}{\Gamma(\delta+n)}\left(\frac{1-t}{e^{i\theta}-t}\right)^n \frac{(1-t)^{\delta-1}}{e^{i\theta}-t}\right| < \frac{kn\,\Gamma(n)}{\Gamma(\delta+n)} < kn^{1-\delta},$$

where k is again a positive number not depending on n and t.[1]) Thus, considering (23), it follows

$$\left|\int_0^\nu \frac{n\Gamma(n)}{\Gamma(\delta+n)}\left(\frac{1-t}{e^{i\theta}-t}\right)^n \frac{(1-t)^{\delta-1}}{e^{i\theta}-t}dt\right| < kn^{1-\delta} \times \nu < kn^{-(\delta-\delta_1)}$$

We therefore need only choose δ_1 less than δ, to see that also the integral over the interval $(0, \nu)$ is zero for $n = \infty$. Thus the whole remainder (11) is zero for $n = \infty$, if only $R(x) > 0$, i.e., since $\lambda = -\infty$ and $\lambda' = 0$, if $R(x) > \lambda'$ and $R(x) > \lambda$. For these values of x the integral

$$\int_0^1 \frac{(1-t)^{x-1}}{e^{i\theta}-t}\,dt$$

can therefore be expanded into a series of factorials; and the theorem of NIELSEN holds in this case.

Again we take the example

$$\varphi(t) = \frac{1}{e^{i\theta}-t} + \frac{1}{(1-t)^\mu}, \quad \begin{pmatrix} 0 < \theta < 2\pi \\ 0 < \mu < 1 \end{pmatrix}.$$

Here $\lambda' = 0$, on account of the first term, and $\lambda = \mu$, on account

[1]) We shall always, in future, denote by k a finite positive number, without always meaning the *same* number by this letter. This will not cause any ambiguity, because the exact value of k is of no importance in our reasonings. For the sake of clearness, however, we shall often mention the quantities on which k does *not* depend.

of the second. If $\varphi(t)$ were equal to the second term only, the integral (1) could be expanded into a series of factorials for $R(x) > \mu$ only, and this series would be *absolutely* converging for these values of x. Thus the *whole* function may also be represented by such a series for $R(x) > \mu$, i.e. for $R(x) > \lambda$ and $R(x) > \lambda'$, but the convergence is, on account of the first term, only *conditional* for $\lambda < R(x) < \lambda' + 1$. This, again, is exactly the proposition of NIELSEN.

6. If, in the first example of the foregoing paragraph, we account for the reason of the validity of this proposition, we infer that it is a consequence of the fact that the expression

$$\left|\frac{1-t}{e^{i\vartheta}-t}\right|^n,$$

for a fixed value of $t > 0$, decreases with $1/n$ as the n-th power of a number less than 1, which causes that, in the integral (11), only an interval has to be considered which, in a proper manner, approaches to zero as n becomes infinite, so that the value of $R(x)$ for which expansion is possible can be depressed by unity. This suggests the idea that something of the kind might occur *as a rule*, if $\varphi(t)$ has $t = 1$ for an *ordinary* point. The truth of this presumption is proved by the following investigation.

We again divide the interval (0,1) of t into two partial intervals, with the point $t = v$ as a common end-point, which is to approach ultimately to zero as n becomes indefinitely large; and we assume, as in the preceding paragraph, for v the value (23). Consider the circle, with centre v and passing through two fixed points C and C' lying on the circumference of the circle of convergence (0,1) of $\varphi(t)$, symmetrically with regard to the axis of real quantities, and in the interior of an arc $D\,A\,D'$ of the latter circle, which does not contain a singular point of $\varphi(t)$, D and D' being also conjugate points, whereas A is the point with the affix $t = 1$. Then, from and after some value of n the value of v will be so small that the circle with centre v does not contain any singular point of $\varphi(t)$ in its interior and on its circumference; and at all points of the latter between the radii $O\,D$ and $O\,D'$, including an arc $E\,B\,E'$ of it (B being the point on that arc with argument zero), the modulus of $\varphi(t)$ will remain under a finite quantity K, independent of n and t. As regards points of the supplementary arc $E\,F\,E'$ of circle (v), F being the point opposite to B, we may remark that $\varphi(t)$ there has a modulus no greater than

$$\bar{\varphi}(1-v''),$$

$\bar{\varphi}(t)$ means the natural majorant of $\varphi(t)$, and ν'' the distance of the points D and E.

We further remark that the radius of the circle (ν) is greater than $1-\nu$, say $1-\nu+\nu'$. It is evident that the numbers ν' and ν'' both approach to zero together with ν, but that their ratios to the latter number remain finite and *different from zero.*

At a point P of the interval $(\nu, 1)$ we have, according to a well-known proposition

$$\left|\frac{\varphi^{(n)}(t)}{n!}\right| < \frac{M}{(1-t+\nu')^n},$$

if M is the greater of the numbers K and $\bar{\varphi}(1-\nu'')$. Instead of this inequality we may write

$$\left|\frac{(1-t)^{n-1}\varphi^{(n)}(t)}{n!}\right| < \left(\frac{1-t}{1-t+\nu'}\right)^{n-1} M\nu'^{-1},$$

or, since for $0 \leq t \leq 1$, $\frac{1-t}{1-t+\nu'} < \frac{1}{1+\nu'}$,

$$\left|\frac{(1-t)^{n-1}\varphi^{(n)}(t)}{n!}\right| < \frac{M\nu'^{-1}}{(1+\nu')^n} \quad \ldots\ldots \quad (24)$$

With regard to $\bar{\varphi}(1-\nu'')$ the following remarks may be made. If, in the equivalence-equation

$$\lim a_n \backsim n^{\lambda'}$$

the quantity λ' is no less than -1, we have, according to the proposition of Cesarò, for any fixed $\delta > 0$

$$\lim_{\nu''=0} (\nu'')^{\lambda'+1+\delta}\,\bar{\varphi}(1-\nu'') = 0,$$

and hence, in virtue of the remark made above on the relation between ν'' and ν,

$$\lim_{\nu=0} \nu^{\lambda'+1+\delta}\,\bar{\varphi}(1-\nu'') = 0$$

and further

$$\lim_{\nu=0} \nu^{\lambda'+1+\delta} \times M = 0$$

since, as a matter of course, the expression $K \times \nu^{\lambda'+1+\delta}$ has, too, zero for its limit.

Thus we may write for (24), in connection with the assumption (23) and the finite, not disappearing ratio between ν and ν'

$$\left|\frac{(1-t)^{n-1}\varphi^{(n)}(t)}{\Gamma(n-1)}\right| < \frac{kn^k}{(1+kn^{\delta_1-1})^n} < kn^k e^{-n^{\delta_1}},$$

where k is again a positive number not depending on n and t. Hence, corresponding to any fixed positive quantity ε chosen arbitrarily small, there is an integral number N, such that the left-hand member of the latter inequality is less in value than ε, for every value of t in the interval $(\nu,1)$, if only $n > N$. For these values of n we have therefore

$$\left| \int_{\nu}^{1} \frac{(1-t)^{x+n-1} \varphi^{(n)}(t)}{\Gamma(x+n)}\, dt \right| < k\varepsilon \int_{\nu}^{1} (1-t)^{R(x)}\, dt$$
$$< k\varepsilon$$

if $R(x) > -1$. For any such value of x, i.e. a fortiori for $R(x) > \lambda'$, since λ' was supposed greater than -1, the part of the integral (11) taken over the interval $(\nu,1)$ has zero for its limit for $n = \infty$.

For the integration over the remaining interval $(0,\nu)$ we apply the mode of treatment of § 3 and the inequality (17). According to the latter there is, corresponding to any fixed δ and ε, chosen as small as we please, an integer N such that we have *uniformly* in the interval $(0,1)$, and hence in $(0,\nu)$,

$$\left| \frac{\varphi^{(n)}(t)\,(1-t)^{\lambda'+\delta+n+1}}{\Gamma(\lambda'+\delta+n+1)} \right| < \varepsilon, \text{ if } n > N.$$

For the interval $(0,\nu)$ it follows from this that, for $n > N$

$$\left| \frac{\varphi^{(n)}(t)\,(1-t)^{x+n-1}}{\Gamma(x+n)} \right| < k\varepsilon\, n^{1-R(x)+\lambda'+\delta}$$

thus

$$\left| \int_{0}^{\nu} \frac{\varphi^{(n)}(t)\,(1-t)^{x+n-1}}{\Gamma(x+n)}\, dt \right| < k\varepsilon\, n^{-R(x)+\lambda'+\delta+\delta_1}$$

If now $R(x) > \lambda'$, we can have chosen the numbers δ and δ_1 so small that $R(x)$ is also greater than $\lambda' + \delta + \delta_1$, and in this case we infer from the latter inequality that the integral over the interval $(0,\nu)$, too, has zero as a limit for $n = \infty$, if $R(x) > \lambda'$. *Thus the theorem of* NIELSEN *has been proved, in case* $t = 0$ *is an ordinary point of the function* $\varphi(t)$.

If a function $\varphi(t)$ has the point $t = 1$ for its only singular point on the circumference of the circle of convergence $(0,1)$, and if, moreover, it satisfies the conditions of HADAMARD, i.e. if it is continuous and "à écart fini" on that circumference, or if a certain derivative of negative order $-\omega$ has this property, then we always have

$$\omega = \lambda = \lambda' + 1,$$

and the theorem of NIELSEN has ceased having anything particular.

Again it may happen that $\varphi(t)$ can be divided into the sum of two functions $\varphi_1(t)$ and $\varphi_2(t)$, the first of which is regular at $t=1$ and the second of which has the latter point as its only singularity on the circumference of the circle (0,1). If, then, the number λ' for $\varphi_1(t)$ is equal to λ_1' and that for $\varphi_2(t)$ to λ_2' and if $\lambda_1' > \lambda_2'$, so that for the whole function $\varphi(t)$ the number λ' is equal to λ_1', the integral (1) can be expanded in a conditionally converging series of factorials for

$$\lambda_2' + 1 < R(x) < \lambda' + 1$$

if $\lambda' = \lambda_1' < \lambda_2' + 1$, and for

$$\lambda' < R(x) < \lambda' + 1$$

if $\lambda_1' > \lambda_2' + 1$. If, in this case, $\varphi_2(t)$ has the properties of HADAMARD, then $\lambda_2' + 1 = \lambda_2 = \lambda$, and the proposition of NIELSEN is valid, which, now, really has a particular meaning.

7. The following proposition is, as a corollary, included in the theorem of the preceding paragraph:

If the coefficients a_n of a function $\varphi(t)$, defined by a power-series

$$\varphi(t) = \sum_0^\infty {}_n\, a_n t^n$$

which has the circle (0.1) *as its domain of convergence, are, for $n=\infty$, equivalent to a power $n^{\lambda'}$ of n, and if the series*

$$\sum_0^\infty {}_n \frac{a_n}{n^{\lambda'+\theta}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

is divergent for $0 < \theta < 1$, the point $t=1$ is a singularity of $\varphi(t)$.

For if $t=1$ is an *ordinary* point of $\varphi(t)$, the series (6), which, except for the factor $\Gamma(x)$, is equal to

$$\sum_0^\infty {}_n \frac{n!\, a_n}{\Gamma(x+n+1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

is convergent for $R(x) > \lambda'$ and the convergence of (25) can be derived from it. For we may write

$$\frac{n!\, a_n}{n^{\lambda'+\theta}} = \frac{n!\, a_n}{\Gamma(x+n+1)} \times \frac{\Gamma(x+n+1)}{\Gamma(n+1)n^{\lambda'+\theta}}$$

If we choose x such that $\lambda' < R(x) < \lambda' + \theta$, the series formed by the first factor, if n takes all values from zero to infinite, converges, as we have already seen, whereas the series, composed of the terms obtained by taking the *first finite differences*, with regard to n, of the second factor, converges *absolutely*; and it is a well-

known truth that the convergence of (25) is a consequence of these two facts. The same thing would hold with regard to the series

$$\Sigma \frac{a_n}{\varphi(n)}$$

if

$$\lim_{n=\infty} \varphi(n) = n^{\lambda'+\theta}, \quad \text{and} \quad \lim_{n=\infty} \Delta\, \varphi(n) = n^{\lambda'+\theta-1}.$$

Therefore, in the statement of the above theorem such a series may be chosen as well. We further remark that λ', which was hitherto supposed to be greater than -1, may also be *less* than the latter number: the theorem of NIELSEN, in the particular case demonstrated in § 6, keeps its validity for those values of λ', though we should have to apply our reasonings to an integral of the form (8) (in a footnote of § 1) in the latter case.

By substituting $t = t' e^{i\varphi}$ in the power-series for $\varphi(t)$ we obtain the more general theorem:

If the coefficients a_n of a power-series in the letter t are equivalent to $n^{\lambda'}$ for $n = \infty$, the function $\varphi(t)$ represented by that series has, on the circumference of its circle of convergence (being the circle (0,1), singularities at all points where the series

$$\Sigma \frac{a_n t^n}{n^{\lambda'+\theta}} \quad (0 < \theta < 1)$$

diverges. We may add that this theorem already holds, if only the *upper limit* of the coefficients a_n is, in the sense of equation (14'), equivalent to $n^{\lambda'}$ for $n = \infty$.

Finitely we observe that the reverse of the proposition does not hold: if the series (25) converges, the point $t = 1$ need not be an ordinary point. To make this clear we need only think of the case that the coefficients a_n differ from zero only for values of n lying at a certain distance from each other; it may happen then that the series (25) converges absolutely, but the function $\varphi(t)$ has its whole circle of convergence as a singular line.

8. As already remarked, we doubt of the *general* validity of NIELSEN's theorem, though we are not in a position to furnish a case of the non-validity. It is our opinion that, if $\lambda' < \lambda < \lambda' + 1$, there will be cases in which the integral (1) cannot, for all values of $R(x) > \lambda$, be expanded into a series of factorials. On the other hand we can prove that such an expansion is not possible for any value of $R(x) < \lambda$, which is a thing not immediately evident if λ lies between λ' and $\lambda' + 1$. [1])

[1]) If $R(x) < \lambda'$, the impossibility is at once evident, since the series-terms have not zero for their limits then.

Suppose the series (25) to converge for $\theta > \theta_1$ and to diverge for $\theta < \theta_1$, and, consequently, the series (26) to converge for $R(x) > \lambda' + \theta_1$, and to diverge for $R(x) < \lambda' + \theta_1$, then the integral (1) will, at any case, not admit an expansion into a series of factorials for any value of $R(x) < \lambda' + \theta_1$. We now shall prove that for any positive δ, taken as small as we please,

$$\lim_{t=1} (1-t)^{\lambda'+\theta_1+\delta} \varphi(t) = 0 \text{ [1]}$$

so that $\lambda \leqq \lambda' + \theta_1$; by this the required proof will have been established.

For the sake of brevity we write

$$\lambda' + \theta_1 + \delta = \alpha.$$

Consider the derivative of negative order $-\alpha$ of $\varphi(t)$, which according to the definition of RIEMANN [2], is given by

$$D^{-\alpha} \varphi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \varphi(u) du = t^\alpha \psi(t),$$

then $\psi(t)$ is a function regular at $t = 0$ with the same circle of convergence (0,1) as $\varphi(t)$ has; its expansion into a power-series is

$$\psi(t) = \sum_0^\infty {}_n \frac{n! a_n t^n}{\Gamma(\alpha+n+1)} \quad \ldots\ldots\ldots \quad (27)$$

From this formula it may be derived that $\psi(t)$ remains finite for $t = 1$, in virtue of the initial hypothesis.

Conversely we have

$$\varphi(t) = D^\alpha [t^\alpha \psi(t)].$$

First, let

$$\lambda' + \theta_1 < 1.$$

Then we may choose δ so small that also $\alpha < 1$ and write

$$\varphi(t) = D^{\alpha-1} D[t^\alpha \psi(t)] \text{ [3]} = D^{\alpha-1} [\alpha t^{\alpha-1} \psi(t) + t^\alpha \psi'(t)] =$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-u)^{-\alpha} [\alpha \psi(u) u^{\alpha-1} + \psi'(u) u^\alpha] du \Bigg\}. \quad (28)$$

Now $\psi(u)$ is, in the range $0 \leqq u \leqq 1$, finite and thus less than a certain number g. Hence

---

[1]) Or for negative values of $\lambda' + \theta_1$, $\lim (1-t)^{\lambda'+\theta_1+n+\delta} \varphi^{(n)}(t) = 0$, if n is such that $\lambda' + \theta_1 + n \geqq 0$.

[2]) See among others BOREL, Leçons sur les séries à termes positifs, p. 75.

[3]) Properly speaking it should be $D \cdot D^{\alpha-1}$, but this operation, in the present case, is equal to $D^{\alpha-1} \cdot D$, since the subject of the operation is *zero* for $t = 0$.

$$\left|\int_0 \psi(u)u^{\alpha-1}(t-u)^{-\alpha}du\right| < g\int_0^t u^{\alpha-1}(t-u)^{-\alpha}\,du$$

or, substituting $u = tv$

$$\left|\int_0^1 \psi(u)u^{\alpha-1}(t-u)^{-\alpha}du\right| < g\int_0^1 v^{\alpha-1}(1-v)^{-\alpha}\,dv,$$

so that the integral in the left-hand member of this inequality remains finite for all values of t in the closed interval (0,1). Further we divide the second integral on the right-hand side of (28) as follows, supposing $t > \frac{1}{2}$,

$$\int_0^t \psi'(u)u^{\alpha}(t-u)^{-\alpha}\,du = \int_0^{t-(1-t)} + \int_{t-(1-t)}^t$$

To the first of these two integrals we apply the second mean value theorem, which is allowed, because the expression $u^{\alpha}(t-u)^{-\alpha}$ increases monotonously in the interval in question. We obtain

$$\int_0^{t-(1-t)} \psi'(u)u^{\alpha}(t-u)^{-\alpha}\,da = (2t-1)^{\alpha}(1-t)^{-\alpha}[\psi(2t-1)-\psi(\beta)]$$

where β is a number in the interval $(0, 2t-1)$. This part of the integral, as $q(t)$ remains within finite limits, is therefore for $t=1$ at most equivalent to $(1-t)^{-\alpha}$. In order to infer the same thing with regard to the second integral, we make use of the fact that

$$\lim_{t=1}(1-t)^n\,\psi^{(n)}(t) = 0, \quad (n = 1, 2, \ldots) \quad . \quad . \quad . \quad . \quad (29)$$

We shall prove this at once; it should not be thought that it is a consequence of the proposition mentioned in a footnote of § 1: it follows solely from the convergence of the series (27) for $t=1$.

If we assume, for a moment, the formula to be true, we have for the whole interval $0 \leq \alpha \leq 1$, if K is a certain positive number, not depending on u,

$$\psi'(u) < \frac{K}{1-u},$$

and so, in the interval of integration $2t-1 \leq u \leq t$

$$\psi'(u) < \frac{K}{1-t},$$

from which it follows

$$\left| \int_{2t-1}^{t} \psi'(u) u^{\alpha} (t-u)^{-\alpha} \, du \right| < \frac{K}{1-t} \int_{2t-1}^{t} (t-u)^{-\alpha} \, du = K(1-t)^{-\alpha},$$

so that this integral, too, is for $t=1$ at most of order $(1-t)^{-\alpha}$. The same holds therefore for the function $\varphi(t)$, and since α may be supposed arbitrarily-little greater than $\lambda' + \theta_1$, we have certainly for every $\delta > 0$

$$\lim_{t=1} (1-t)^{\lambda'+\theta_1+\delta} \varphi(t) = 0$$

and thus, as we proposed to show

$$\lambda \leqq \lambda' + \theta_1.$$

Secondly, let $\lambda' + \theta_1$ lie between the integers $p-1$ and p; we may choose δ so small that the same holds for $\lambda' + \theta_1 + \delta = \alpha$. We write

$$\alpha = p-1+\alpha' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

so that

$$0 < \alpha' < 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

In this case we have the following reduction

$$\varphi(t) = D^{\alpha'-1} D^{p} [t^{\alpha} \psi(t)] \text{ [1]}$$

$$= D^{\alpha'-1} \left[\Gamma(\alpha'+p) \sum_{0}^{p} \frac{p_m t^{\alpha'+m-1} \psi^{(m)}(t)}{\Gamma(\alpha'+m)} \right]$$

Owing to (31) we may, as in the former case, using here the inequality (29) for $n=m$, prove that the expression

$$D^{\alpha'-1} [t^{\alpha'+m-1} \psi^{(m)}(t)]$$

is at most equivalent to $(1-t)^{-(\alpha'+m-1)}$ and thus $\varphi(t)$, as m is no greater than p, is of an order no higher than that of $(1-t)^{-(\alpha'+p-1)}$, that is, according to (30), of the order $(1-t)^{-\alpha}$. Thus the required result is obtained completely.

9. We now give a proof of the proposition used in the preceding paragraph. It may be stated as follows:

If the expansion in a power-series of a function $\varphi(t)$ converges at the point $t=1$ of the circle of convergence (0,1), we have for all positive integral values of n

$$\lim_{t=1} (1-t)^{n} \varphi^{(n)}(t) = 0 \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

This proposition, of course, ceases to have a particular meaning,

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[1]) Properly speaking it should be $D^{p} D^{\alpha'-1}$, but this comes to the same thing as $D^{\alpha'-1} D^{p}$, because $t=(p-1)$-times the subject of operation is zero for $t=0$.

if $t=1$ is not a singular point of $\varphi(t)$, but if it is, the proposition is not a matter of course.

Since, if the coefficients of the power-series in question are complex, the two series formed separately by means of the real and of the imaginary parts of those coefficients must both converge, we may without loss of generality suppose the coefficients to be real quantities. We then consider, together with the function

$$\varphi(t) = a_0 + a_1 t + \ldots + a_n t^n + \ldots$$

the function

$$f(t) = \frac{\varphi(t)}{1-t} = \sum_0^\infty{}_n \, s_n t^n \quad \ldots\ldots\ldots \quad (33)$$

where

$$s_n = \sum_0^n{}_p \, a_p,$$

Since s_n, as n becomes indefinitely large, approaches to a definite limit s, the series (33) behaves, so far as regards its terms for large values of n, as the power-series of the function

$$\frac{s}{1-t},$$

and according to the reasoning of Cesaro we have not only

$$\lim_{t=1}\left[f(t) : \frac{1}{1-t} \right] = s,$$

but also

$$\lim_{t=1}\left[f^{(n)}(t) : \frac{n!}{(1-t)^{n+1}} \right] = s. \quad \ldots\ldots\ldots \quad (34)$$

Further, from n-fold differentiation of the identity

$$\varphi(t = (1-t) f(t)$$

we obtain the new one

$$\frac{(1-t)^n \varphi^{(n)}(t)}{n!} = \frac{(1-t)^{n+1} f^{(n)}(t)}{n!} - \frac{(1-t)^n f^{(n-1)}(t)}{(n-1)!}.$$

The limit of the right-hand side of the latter equation for $t=1$, is, by (34), equal to zero for all positive integral n-values, and the required formula (32) has thus been proved.

By substituting $t = t' e^{i\varphi}$ we obtain: *If the expansion in a power-series of a function $\varphi(t)$ converges at the point $t = e^{i\varphi}$ of its circle of convergence (0,1), then, for all positive integral values of n and for real values of t', we have*

$$\lim_{t'=1} (1-t')^n \varphi^{(n)}(t' e^{i\varphi}) = 0$$

Physics. - *"Deduction of the third virial coefficient for material points (eventually for rigid spheres), which exert central forces on each other"*. By Prof. W. H. KEESOM and Mrs. C. NORDSTRÖM-VAN LEEUWEN. (Communication N°. 3*a* from the Laboratory of Physics and Physical Chemistry of the Veterinary College at Utrecht). (Communicated by Prof. H. KAMERLINGH ONNES).

(Communicated in the meeting of September 29, 1918).

§ 1. *Introduction.* This paper contains a continuation of the research, started in Supplement N°. 24*a* of the communications from the physical laboratory at Leiden (These Proceed., June 1912), in which the existing data on the virial coefficients B and C in KAMERLINGH ONNES' [1]) empirical equation of state [2])

$$pv = A\left\{1 + \frac{B}{v} + \frac{C}{v^2} + \frac{D}{v^4} + \frac{E}{v^6} + \frac{F}{v^8}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

were compared with formulae, which can be derived for those virial coefficients starting from different hypotheses on the structure and mutual actions of the molecules.

In the preceding [3]) papers on this subject especially the second virial coefficient B was considered. In the present communication the third virial coefficient C will be treated. To that purpose there will be derived in § 2 a general expression for this third virial-coefficient for material points (eventually rigid spheres) which exert central forces on each other. In § 3 C is calculated for the case that the molecules (rigid spheres) do not attract each other. The thus obtained value is at the same time the first term of a series of ascending powers of T^{-1}, in which C can be developed for the case that the attraction between the molecules varies with the distance according to a law $r^{-(q+1)}$ (potential energy proportional to r^{-q}). In Comm. N°. 3*b* the development will be given for the next two terms of this series for the laws of force r^{-5} and r^{-6}.

These calculations had already been started, when the importance

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[1]) H. KAMERLINGH ONNES. Comm. Leiden N°. 71; These Proceed. June 1901. Comm. N°. 74; Arch. Néerl. (2) 6 (1901), p. 874.

[2]) Comp. H. KAMERLINGH ONNES and W. H. KEESOM. Die Zustandsgleichung. Math. Enz. V 10, Leiden Suppl. N°. 23 (1912), § 36.

[3]) W. H. KEESOM. Comm. Leiden Suppl. N°. 24*b* (These Proc. June 1912), 25 (Sept. 1912), 26 (Oct. 1912), 39*a* (Sept. 1915), 39*b* (Oct. 1915), W. H. KEESOM and Miss C. VAN LEEUWEN, Leiden Suppl. N°. 39*c* (March 1916).

of the theoretical investigation of C was increased by a remark of HOLST [1]), who concluded from the investigations on the compressibility, that for substances as water and ammonia the molecules of which possess an electric moment such as that of a doublet, C is negative in the considered temperature interval, while for normal substances C is found positive. A calculation of C for molecules whose attraction is equivalent to that of a doublet at their centre and a comparison of this result e.g. with that for molecules like those considered in this communication or with that for quadruplets will have to prove whether really the deviating behaviour of C for the substances mentioned must be ascribed to the molecule possessing an electric moment such as that of a doublet. The calculation of C for doublets and quadruplets must however be postponed till later.

§ 2. *Deduction of a general expression for the third virial coefficient for material points which exert central forces on each other.* This deduction is analogous to that of the second virial coefficient for this case given in Suppl. Leiden N°. 24*b* § 5, which paper may be referred to with respect to the method and the notations already used there.

Definition of the macro-complexion.

Generalizing (23) l.c., the n_{11} molecules in $dv_1 dw_1$ will now be distinguished in:

n_{11a} single molecules (with no other molecule within their sphere of action),

n_{11b_1} molecules belonging to pairs with a mutual distance between r_1 and $r_1 + dr_1$,

n_{11b_2} molecules belonging to pairs with a mutual distance between r_2 and $r_2 + dr_2$

etc.

$n_{11c_{12}}$ molecules belonging to sets of three, the mutual distances of which are resp. lying between r_1 and $r_1 + dr_1$, r_2 and $r_2 + dr_2$, r_3 and $r_3 + dr_3$,

etc. (2)

We define the group macro-complexion [2]) by giving the numbers $n_{11a} \ldots$ without attending to any individuality of the molecules.

The individual macro-complexion [2]) is defined as follows:

n_{11a} definitely indicated molecules are single,

. . . . . .

[1]) G. HOLST. Leiden Suppl. N°. 41*f*. These Proceed. Jan. 1917, HOLST points there also to the importance of the knowledge of the behaviour of C for the determination of the molecular weight from the gas density, comp. also H. KAMERLINGH ONNES and W. H. KEESOM. Die Zustandsgleichung. Leiden Suppl. N°. 23 §§ 77—80.

[2]) Comp. Leiden Suppl. N°. 24*a* § 2 (These Proceed. June 1912).

n_{11c123} definitely indicated molecules belong to sets of three, the mutual distances of which are r_1 (dr_1), r_2 (dr_2), r_3 (dr_3), etc.

Here we have not yet fixed in what way these n_{11c123} molecules are divided into sets of three, and which two molecules of each set of three have the distance r_1 (dr_1) and which that r_2 (dr_2).

The number of individual macro-complexions contained in the group macro-complexion is:

$$\frac{n!}{n_{11a}!\ n_{11b1}! \ldots . n_{11c123}! \ldots .} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The micro-complexion is defined in the way mentioned Suppl. Leiden N°. 24*b* § 5.

Number of micro-complexions in the individual macro-complexion:

In dv_1 n_1 molecules are to be placed. First we place the n_{1a} single molecules. There are at the disposal of the

1st molecule: $\varkappa$ places

2nd ,, $\varkappa\left\{1-\frac{b}{dv_1}\right\}$,, ., where

$$b = \tfrac{4}{3}\pi r^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

has been written for the volume of the sphere of action,

3rd molecule: $\varkappa\left\{1-2\frac{b-\beta}{dv_1}\right\}$ places:

(here an amount β has been subtracted from b because of the occurrence of a certain number of cases in which the spheres of action of the molecules 1 and 2 partially overlap each other);

4th molecule: $\varkappa\left\{1-3\frac{b-2\beta}{dv_1}\right\}$ places, etc.

Thus the distribution of the n_{1a} molecules gives for the number in question the factor:

$$\varkappa^{n_{1a}}\left\{1-\frac{b}{dv_1}\right\}\left\{1-2\frac{b-\beta}{dv_1}\right\}\left\{1-3\frac{b-2\beta}{dv_1}\right\}\ldots\left\{1-(n_{1a}-1)\frac{b-(n_{1a}-2)\beta}{dv_1}\right\}. \quad (5)$$

Calculation of β: Call the hatched spherical segment $s(r_{12})$.

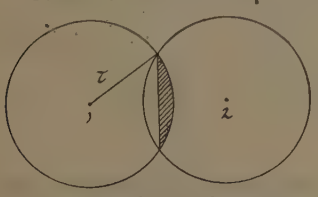

Fig. 1.

Probability that molecule 2 has a distance r_{12} from 1:

$$\frac{4\pi r^2_{12} dr_{12}}{dv_1}.$$

From this we find:

$$\beta = \int_{r}^{2r} \frac{4\pi r^2_{12} dr_{12}}{dv_1} s(r_{12}).$$

As

$$s(r_{12}) = \frac{2\pi}{3}\{\tau^3 - \tfrac{3}{4}\tau^2 r_{12} + \tfrac{1}{16} r^3{}_{12}\},$$

we get

$$\beta = \tfrac{17}{64}\frac{b^2}{dv_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Then we place the n_{1b1} molecules:

At the disposal of the first molecule are:

$$\varkappa\left\{1 - n_{1a}\frac{b}{dv_1}\right\} \text{ places,}$$

where a term with β has been omitted, being negligible for the order of magnitude wanted;

of the 2nd molecule:

here we must take into consideration that of the space at the disposal of this molecule in the mean a certain part is occupied by one of the n_{1a} molecules; doing this we find for the number of places at the disposal of the 2nd molecule:

$$\varkappa\frac{4\pi r_1{}^2 dr_1}{dv_1}\left\{1 - n_{1a}\frac{\pi r_1(\tau^2 - {}^1/_{12} r_1{}^2)}{dv_1}\right\} \text{ places;}$$

of the 3rd molecule:

$$\varkappa\left\{1 - (n_{1a}+2)\frac{b}{dv_1}\right\} \text{ places,}$$

etc.

Thus the placing of all the n_{1b} molecules (comp. Suppl. Leiden N°. 24*b* § 5) gives us the factor:

$$\varkappa^{n_{1b}} \prod_{dr} \frac{n_{1b1}!}{2^{\frac{n_{1b1}}{2}}\left(\frac{n_{1b1}}{2}\right)!}\left\{1 - n_{1a}\frac{b}{dv_1}\right\}\left\{1 - (n_{1a}+2)\frac{b}{dv_1}\right\}\ldots\ldots \left(\frac{4\pi r_1{}^2 dr_1}{dv_1}\right)^{\frac{n_{1b1}}{2}}\left\{1 - n_{1a}\frac{\pi r_1(\tau^2 - {}^1/_{12} r_1{}^2)}{dv_1}\right\}^{\frac{n_{1b1}}{2}} \quad (7)$$

Now we are going to place the n_{1c123} definitely fixed molecules, belonging to sets of three, with the mutual distances r_1 (dr_1), r_2 (dr_2), r_3 (dr_3). In $\frac{n_{1c123}!}{\left(\frac{n_{1c123}}{3}\right)!}$ ways these molecules can be combined to such sets, where it has been taken into consideration that 3 fixed molecules can form such a set in 3! different ways viz. 1st with distance from molecule 1 to molecule $2 = r_1$, distance 2 to $3 = r_3$, distance 1 to $3 = r_2$; 2nd distance 1 to $2 = r_1$, distance 2 to $3 = r_3$, distance 1 to 3 r_2; etc.

Now we calculate the number of places at the disposal of a definite set of three, going only until the required order of magnitude:

molecule 1: $\varkappa$ places

molecule 2: $\varkappa \dfrac{4\pi r_1^2 dr_1}{dv_1}$,,

molecule 3: at the disposal of this molecule is the space described by the hatched parallelogram in revolving about the line 1, 2.

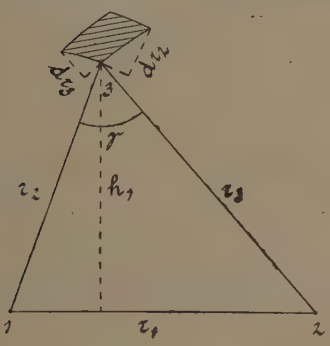

Fig. 2.

Contents: $\dfrac{dr_2 dr_3}{\sin\gamma} \cdot 2\pi h_1$.

The number of places in question is therefore $\varkappa . 2\pi \dfrac{r_2 r_3}{r_1} \dfrac{dr_2 dr_3}{dv_1}$, where the relation

$$r_1 h_1 = r_2 r_3 \sin\gamma$$

has been used.

The n_{1c} molecules thus give the factor:

$$\varkappa^{n_{1c}} \prod_{dr_1 dr_2 dr_3} \frac{n_{1c123}!}{\left(\frac{n_{1c123}}{3}\right)!} \left(\frac{8\pi^2 r_1 r_2 r_3 \, dr_1 dr_2 dr_3}{dv_1^2}\right)^{\frac{n_{1c123}}{3}} \quad . \quad . \quad . \quad . \quad (8)$$

where the sign of multiplication has to be extended over all possible combinations r_1, r_2, r_3.

The number of micro-complexions in the group macro-complexion, W, is now obtained by multiplication of (3) by the product of the factors (5), (7), and (8), taken for each of the volume elements dv_1, dv_2, etc.

From this we derive

$$\begin{aligned}
\ln W = &- n_{11a} \ln n_{11a} \cdot \ldots - n_{11b1} \ln n_{11b1} \cdot \ldots - n_{11c123} \ln n_{11c123} \\
&\ldots - \sum_{dv} \tfrac{1}{2} n^2_{1a} \frac{b}{dv_1} - \sum_{dv} \tfrac{1}{2} n_{1a} n_{1b} \frac{b}{dv_1} + \sum_{dv} \tfrac{1}{3} n^3_{1a} \frac{\beta}{dv_1} - \\
&- \sum_{dv} \tfrac{1}{6} n^3_{1a} \frac{b^2}{dv_1^2} + \sum_{dv} \sum_{dr} \left[\frac{n_{1b1}}{2} \ln n_{1b1} - \frac{n_{1b1}}{2} + \right. \\
&\left. + \frac{n_{1b1}}{2} \ln \frac{4\pi r_1^2 dr_1}{dv_1} - \frac{n_{1b1}}{2} \frac{n_{1a}}{dv_1} \pi r_1 (\tau^2 - \tfrac{1}{12} r^2_1) \right] + \\
&+ \sum_{dv} \sum_{dr_1 dr_2 dr_3} \left[\tfrac{2}{3} n_{1c123} \ln n_{1c123} - \tfrac{2}{3} n_{1c123} + \right. \\
&\left. + \frac{n_{1c123}}{3} \ln \frac{24\pi^2 r_1 r_2 r_3 \, dr_1 dr_2 dr_3}{dv_1^2} \right].
\end{aligned} \qquad (9)$$

The state of equilibrium:

As in Suppl. Leiden N°. 24*b* § 5: $-\varphi(r_1)$ may represent the potential energy with respect to the mutual forces for a pair of molecules with the mutual distance r_1, where for $r_1 > \tau$ $\varphi(r_1) = 0$. For a set of three molecules the total potential energy is:

$$-\Phi(r_1, r_2, r_3) = -\{\varphi(r_1) + \varphi(r_2) + \varphi(r_3)\}, \quad . \quad . \quad . \quad (10)$$

where we assume, that the attraction between two molecules is not changed by the presence of a third molecule.

The condition for the energy is then:

$$u = \sum_{dv}\sum_{dw} n_{11} u_{w1} - \tfrac{1}{2}\sum_{dv}\sum_{dr} n_{1b1}\varphi(r_1) - \tfrac{1}{3}\sum_{dv}\sum_{dr_1 dr_2 dr_3} n_{1c123}\Phi(r_1, r_2, r_3) = const. \quad (11)$$

For the state of equilibrium we then find after some reductions e.g. for the distribution of the molecules into single molecules, into molecules belonging to pairs, into molecules belonging to sets of three, with the required degree of approximation:

$$\left.\begin{aligned} n_a &= n\left\{1 - \frac{n}{v}P + \frac{n^2}{v^2}(-bP + 2P^2 + R - S)\right\}, \\ n_b &= \frac{n^2}{v}\left\{P + \frac{n}{v}(bP - 2P^2 - R)\right\}, \\ n_c &= \frac{n^3}{v^2}S, \end{aligned}\right\} \quad (12)$$

where passing from summations to integrals:

$$\left.\begin{aligned} P &= \int_0^\tau e^{h\varphi(r)}\, 4\pi r^2 dr, \\ R &= \int_0^\tau e^{h\varphi(r)}(\tau^2 - {}^1/_{12}\, r^2)\, 4\pi^2 r^3\, dr, \\ S &= \iiint e^{h\Phi(r_1, r_2, r_3)}\, 24\pi^2 r_1 r_2 r_3\, dr_1 dr_2 dr_3. \end{aligned}\right\} \quad (13)$$

Deducing further the entropy, then the free energy ψ and from this the pressure, we finally find for the second virial coefficient:

$$B = \tfrac{1}{2}n(b - P), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

analogous with Suppl. Leiden N°. 24*b* equation (40), and for the third:

$$C = \tfrac{1}{3}n^2\left(\tfrac{1}{3}\tfrac{5}{2}b^2 - 3bP + 3P^2 + 3R - 2S\right) \quad . \quad . \quad . \quad . \quad (15)$$

§ 3. *Development for rigid spheres without attraction.* As first

case for the development of C according to (15) and (13) we put

$$\varphi(r) = -\infty \quad \text{for} \quad r < \sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$\varphi(r) = 0 \quad \text{for} \quad \tau > r > \sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

with which case we have to do when we consider rigid spheres with diameter σ which do not attract each other. We must then obtain the same result found for the first time in another way by JÄGER [1]) as a second volume correction in VAN DER WAALS' equation of state. Notwithstanding this the calculation based upon the general expression (15) we have derived above, will be discussed here rather in details, because the term obtained here may be considered as the first of the development of C in ascending powers of h for $\varphi(r) = cr^{-q}$. The calculation of the following in the next communication may then be discussed more shortly.

As also for $r > \tau$ $\varphi(r) = 0$, the quantity τ must evidently vanish from the result.

We easily find that for $\varphi(r)$ according to (16) and (17):

$$\left.\begin{aligned} P_1 &= \tfrac{4}{3}\pi(\tau^3 - \sigma^3) \\ R_1 &= \tfrac{17}{18}\pi^2\tau^6 - \pi^2\tau^3\sigma^4 + \tfrac{1}{18}\pi^2\sigma^6 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where the quantities P and R have been marked with an index 1 in order to indicate that these values are the first terms in the development in ascending powers of h for $\varphi(r) = cr^{-q}$.

For the calculation of $S_1 = \iiint 24\pi^2 r_1 r_2 r_3 \, dr_1 dr_2 dr_3$ it may be useful that $2\pi \iint \frac{r_2 r_3}{r_1} dr_2 dr_3$ represents the space at the disposal of molecule 3, when of the set of three the molecules 1 and 2 have already taken their places.

The mutual distances may be thus numbered that

$$r_1 \geqq r_2 \geqq r_3.$$

Now two parts of the domain of integration must be distinguished:

1. $2\tau > r_1 > \tau$.

The sets of three belonging to this part must still be subdivided into:

a. $r_1 - \tau > \sigma$,

b. $r_1 - \tau < \sigma$.

We suppose namely $\tau > 2\sigma$, so that the case $r_1 < 2\sigma$ need not be considered here.

Let us consider in details case *a*:

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[1]) G. JÄGER. Wien Sitz.-Ber. [2*a*] 105 (1896), p. 15, 97. For further literature see H. KAMERLINGH ONNES and W. H. KEESOM. Die Zustandsgleichung § 40*a*.

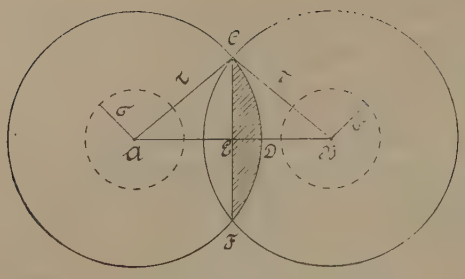

Fig. 3.

In Fig. 3, where A and B are the centres of the molecules 1 and 2, so that $AB = r_1$, the hatched spherical segment represents the space disposable for molecule 3. The volume of this segment is:

$$\tfrac{2}{3}\pi(\tau^3 - \tfrac{3}{4}\tau^2 r_1 + \tfrac{1}{16} r_1^3).$$

Multiplication by $12\pi r_1^2 dr_1$ and integration between $\tau + \sigma$ and 2τ gives for the contribution to S_1:

$$\frac{\pi^2}{12}\{17\tau^6 - 30\tau^5\sigma - 3\tau^4\sigma^2 + 20\tau^3\sigma^3 + 3\tau^2\sigma^4 - 6\tau\sigma^5 - \sigma^6\}.$$

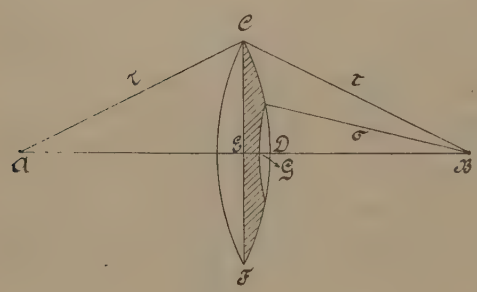

Fig. 4.

b. Now not the whole spherical segment $CDFE$ of Fig. 3 is at the disposal of molecule 3, but only the part of it that is not overlapped by the sphere of distance of molecule 2 (sphere BG), which part has been represented hatched in fig. 4[1]). After integration with respect to r_1 between τ and $\tau + \sigma$ this gives:

$$\frac{\pi^2}{12}\{30\tau^5\sigma + 3\tau^4\sigma^2 - 20\tau^3\sigma^3 - 39\tau^2\sigma^4 + 6\tau\sigma^5 + 3\sigma^6\}.$$

2. $\tau > r_1 > \sigma$.

This part too must again be subdivided, this time into:

a. $\tau > r_1 > 2\sigma$,

b. $2\sigma > r_1 > \sigma$.

a. The space disposable for molecule 3 (the hatched part in Fig. 5) is now bounded by: sphere AB with radius r_1 (as $r_2 \leqq r_1$), plane CDG (as $r_3 \leqq r_2$), and sphere BF with radius σ (sphere of distance of molecule 2). Contribution to S_1 after integration with respect to r_1 between 2σ and τ:

$$\frac{\pi^2}{12}\{5\tau^6 - 32\tau^5\sigma^3 + 18\tau^2\sigma^4 - 136\sigma^6\}.$$

b. Space at the disposal of molecule 3 bounded in the same way as sub *a*; now sphere BF cuts, however, the plane CDF, so that only an annular space (hatched) is left.

[1]) In drawing this figure the supposition $\tau < 2\sigma$ has been neglected.

Contribution to S_1 after integration between σ and 2σ:

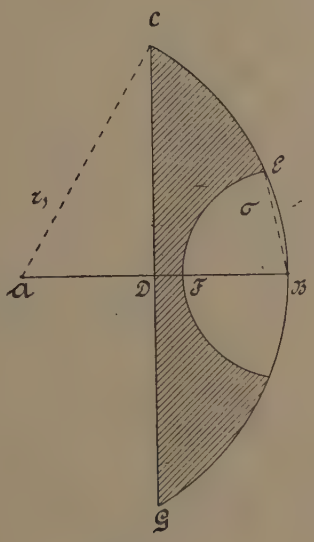

Fig. 5.

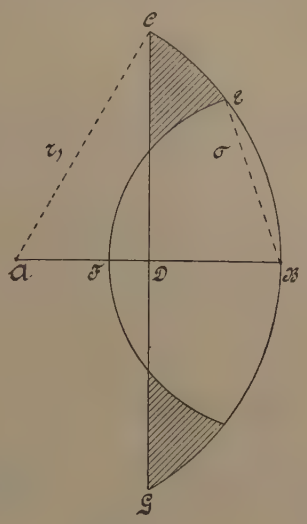

Fig. 6.

$$\frac{\pi^2}{12} \cdot 162\sigma^6.$$

Adding the four contributions we find:

$$S_1 = \frac{\pi^2}{6}\{11\tau^6 - 16\tau^3\sigma^3 - 9\tau^2\sigma^4 + 14\sigma^6\}. \quad . \quad . \quad . \quad . \quad (19)$$

From (14) we see that (ascribing to the index 1 the same significance as above):

$$\left.\begin{aligned} C_1 &= \tfrac{1}{8} n^2 \cdot \tfrac{5}{6} \pi^2\sigma^6 \\ &= \tfrac{5}{32} n^2 \cdot (\tfrac{4}{3}\pi\sigma^3)^2. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

This result agrees with that first found by JÄGER l. c. Introducing namely the quantity b_w from the equation of state of VAN DER WAALS:

$$b_w = n \cdot \tfrac{1}{2} \cdot \tfrac{4}{3}\pi\sigma^3,$$

we find for (20): $C_1 = \frac{5}{8} b_w^2$, the agreement of which result with that of JÄGER is e. g. evident by comparison with Die Zustandsgleichung l. c. § 40*a*.

Physics. — *"Development of the third virial coefficient for material points (eventually rigid spheres), which exert central attracting forces on each other proportional with r^{-5} or r^{-6}."* By Prof. W. H. KEESOM and Mrs C. NORDSTRÖM-VAN LEEUWEN. (Communication N°. 3*b* from the Laboratory of Physics and Physical Chemistry of the Veterinary College at Utrecht). (Communicated by Prof. H. KAMERLINGH ONNES).

(Communicated in the meeting of September 29, 1918).

§ 1. In Comm. N°. 3*a* a general expression was found for the third virial coefficient for material points (eventually rigid spheres), which exert central forces on each other, while as a special case the value of C was derived from that formula for spheres without attraction. As has been remarked already there the value of C found for this case may at the same time be considered as the first term in the development of C according to ascending powers of T^{-1} for a law of force $r^{-(q+1)}$ (potential energy proportional with r^{-q}). In this communication some of the next terms of that development into a series will be calculated for $q=4$ (§ 2) and $q=5$ (§ 3).

§ 2. *Development for rigid spheres with attraction proportional with* r^{-5}. Instead of (17) of the preceding communication we now put:

$$\varphi(r) = \frac{c}{r^4} \quad \text{for} \quad \tau > r > \sigma, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)^1)$$

and develop the exponential factor in P, R, and S according to (13) according to ascending powers of h, indicating the succeeding terms by indices.

In the same way C splits up into:

$$C = C_1 + C_2 + C_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

C_1 has been calculated already in Comm. N°. 3*a* § 3, see equ. (20). With a view to C_2 we must calculate:

$$S_2 = 24\,\pi^2\,hc \iiint \left\{ \frac{r_1\,r_2}{r_2^3} + \frac{r_1\,r_3}{r_2^3} + \frac{r_2\,r_3}{r_1^3} \right\} dr_1\,dr_2\,dr_3.$$

The domain of integration is divided in the same way as in § 3

[1]) The equations have been numbered in continuation of those of Comm. N°. 3*a*.

of Comm. N°. 3*a*. For the domain sub 1, for which $r_1 > \tau$ and thus $\varphi(r_1) = 0$, the last term in the brackets of S_2 must be omitted.

As to the integrations with respect to r_3 and r_2 in the domain 1*b* we must still discern between:

$$\alpha: \quad \tau > r_2 > r_1 - \sigma$$

and

$$\beta: \quad r_1 - \sigma > r_2 > \frac{r_1}{2},$$

in the same way in 2*a* between:

$$\alpha: \quad r_1 > r_2 > r_1 - \sigma$$

and

$$\beta: \quad r_1 - \sigma > r_2 > \frac{r_1}{2}.$$

In the following table the different integration limits have been collected for the succeeding integrations with respect to r_3, r_2, and r_1:

Integration with respect to	1			2		
	a	*b*		*a*		*b*
		α	β	α	β	
r_3	$r_1 - r_2$, r_2	σ, r_2	$r_1 - r_2$, r_2	σ, r_2	$r_1 - r_2$, r_2	σ, r_2
r_2	$\frac{r_1}{2}$, τ	$r_1 - \sigma$, τ	$\frac{r_1}{2}$, $r_1 - \tau$	$r_1 - \sigma$, r_1	$\frac{r_1}{2}$, $r_1 - \sigma$	σ, r_1
r_1	$\tau + \sigma$, 2τ	τ, $\tau + \sigma$		2σ, τ		σ, 2σ

Finally we find:

$$S_2 = \pi^2 hc \left\{ 8\frac{\tau^3}{\sigma} - \tfrac{33}{4}\tau^2 - (11 + 6\,ln\,2)\,\sigma^2 + 16\frac{\sigma^3}{\tau} + 6\tau^2 ln\frac{\tau}{\sigma} \right\} \quad . \quad . \quad (23)$$

Further we have

$$\left.\begin{aligned} P_2 &= 4\pi hc \left\{ \frac{1}{\sigma} - \frac{1}{\tau} \right\} \\ R_2 &= \pi^2 hc \left\{ 4\tau^2 ln\frac{\tau}{\sigma} - \tfrac{1}{6}\tau^2 + \tfrac{1}{6}\sigma^2 \right\} \end{aligned}\right\} \quad . \quad . \quad . \quad (24)$$

Using the value of P_1 found in (18) we find:

$$C_2 = \tfrac{1}{3} n^2 . \pi^2 hc\,(12\,ln\,2 - \tfrac{19}{2})\,\sigma^2 \quad . \quad . \quad . \quad . \quad . \quad (25)$$

C_3 follows from

$$S_3 = 12\,\pi^2 h^3 c^3 \iiint \left\{ \frac{r_1 r_2}{r_3^7} + \frac{r_1 r_3}{r_2^7} + \frac{r_2 r_3}{r_1^7} + \frac{2r_1}{r_2^3 r_3^3} + \frac{2r_2}{r_1^3 r_3^3} + \frac{2r_3}{r_1^3 r_2^3} \right\} dr_1 dr_2 dr_3,$$

where for domain 1 the 3rd, 5th, and 6th terms between the brackets must be omitted. Performing the calculation in the same way as for C_2 we find:

$$S_3 = \pi^2 h^3 c^3 \left\{ \tfrac{4}{5}\frac{\tau^5}{\sigma^5} + \tfrac{3}{4}\frac{\tau^4}{\sigma^4} + (\tfrac{9581}{3360} - 12\ln 2)\frac{1}{\sigma^3} - \frac{66}{\sigma\tau} + \tfrac{5443}{4060}\frac{1}{\tau^3} - \right.$$
$$\left. \tfrac{3}{2}\frac{\sigma^3}{\tau^4} + \tfrac{3}{5}\frac{\sigma^5}{\tau^5} + \frac{18}{\sigma^3}\ln\frac{\tau}{\tau-\sigma} - \frac{6}{\tau^3}\ln\frac{\tau}{\tau-\sigma} \right\} \quad (26)$$

while

$$P_3 = \tfrac{3}{5}\pi h^3 c^3 \left\{ \frac{1}{\sigma^5} - \frac{1}{\tau^5} \right\}$$
$$R_3 = \pi^2 h^3 c^3 \left\{ \tfrac{1}{2}\frac{\tau^3}{\sigma^4} - \tfrac{1}{12}\frac{1}{\sigma^3} - \tfrac{5}{12}\frac{1}{\tau^3} \right\} \quad \ldots \quad (27)$$

so that finally

$$C_3 = \tfrac{1}{3}n^2 \cdot \pi^2 h^3 c^3 \left\{ (24\ln 2 - \tfrac{24533}{1680})\frac{1}{\sigma^2} + \frac{36}{\sigma\tau} + \frac{18}{\tau^3} + \right.$$
$$\left. + \frac{3\sigma^3}{\tau^4} - \frac{36}{\sigma^3}\ln\frac{\tau}{\tau-\sigma} + \frac{12}{\tau^3}\ln\frac{\tau}{\tau-\sigma} \right\} \quad \ldots \quad (28)$$

Remarkable is that the radius τ of the sphere of action does no longer occur in C_2. Developing the logarithmic terms in (28) according to ascending powers of $\frac{\sigma}{\tau}$ we obtain

$$C_3 = \tfrac{1}{3}n^2 . \pi^2 h^3 c^3 \frac{1}{\sigma^2}\{(24\ln 2 - \tfrac{24533}{1680}) - \tfrac{16}{5}\frac{\sigma^3}{\tau^5} \ldots\} \quad \ldots \quad (29)$$

so that evidently the attractive forces excited by the molecules on each other over mutual distances greater than a certain distance τ, furnish a contribution to C_3, the ratio of which to the total term is of the same order of magnitude as that in which the forces decrease at an increase of the mutual distance from σ to τ.

If now we put $\tau = \infty$, we find, adding (20), (25) and (29) and introducing the potential energy at contact

$$v = \frac{c}{\sigma^4},$$

$$C = \tfrac{5}{32} \cdot n^2 (\tfrac{4}{3}\pi\sigma^3)^2 \{1 - \tfrac{3}{5}(19 - 24\ln 2) hv + \tfrac{3}{400}(3840\ln 2 - 2453)(hv)^2 \ldots\}$$

or

$$C = \tfrac{5}{32} n^2 (\tfrac{4}{3}\pi\sigma^3)^2 \{1 - 1{,}418\, hv + 1{,}566\,(hv)^2 \ldots\} \quad \ldots \quad (30)$$

As $h = \frac{1}{k_P T}$, k_P being the well known constant of PLANCK, this gives us the first term in the development of C according to ascending powers of T^{-1}.

§ 3. *Development for rigid spheres with attraction proportional with* r^{-6}. Performing the calculation as in § 2, but now with

$$\varphi(r) = \frac{c}{r^6} \quad \text{for} \quad \tau > r > \sigma. \quad . \quad . \quad . \quad (31)$$

we find:

$$\left.\begin{aligned} C_2 &= -\tfrac{1}{3} n^2 . \pi^2 h c \sigma \\ C_3 &= \tfrac{1}{3} n^2 \pi^2 h^2 c^2 \left\{ (\tfrac{763399}{26888} - 40 \ln 2) \frac{1}{\sigma^4} - \frac{32}{\sigma^3 \tau} - \frac{12}{\sigma^2 \tau^2} - \right. \\ &\left. - \frac{16}{3\sigma\tau^3} - \frac{2}{\tau^4} + \tfrac{4}{5} \frac{\sigma^5}{\tau^6} - \frac{8}{\sigma^3(\tau-\sigma)} - \frac{40}{\sigma^4} \ln \frac{\tau-\sigma}{\tau} \right\} \end{aligned}\right\} . \quad (32)$$

The radius of the sphere of action neither occurs here in C_2, while the development of C_3 according to ascending powers of $\frac{\sigma}{\tau}$ induces the same remark as was made in § 2 concerning the contribution furnished by the attractive forces for greater mutual distances.

For $\tau = \infty$ we obtain:

$$\left.\begin{aligned} &C = \tfrac{5}{3\sqrt{2}} n^2 (\tfrac{4}{3}\pi\sigma^3)^2 \{1 - \tfrac{6}{5} hv + (\tfrac{763399}{22400} - 48 \ln 2)(hv)^2 .. \} \\ &\text{or} \\ &C = \tfrac{5}{3\sqrt{2}} n^2 (\tfrac{4}{3}\pi\sigma^3)^2 \{1 - 1{,}2\, hv + 0{,}809\, (hv)^2 \ \ ..\} \end{aligned}\right\} . \quad . \quad (33)$$

§ 4. In order to represent the dependency of C on the temperature more in details and to compare this for the two laws of force discussed in this paper, we shall introduce a temperature characteristic of each gas as a reduction temperature [1]). As the series for B found Suppl. Leiden N°. 24*b* § 3 eq. (42) converges still sufficiently rapidly at the BOYLE-point, this point suggests itself as reduction temperature.

For $\varphi(r) = cr^{-4}$ equation (42) Suppl. Leiden N°. 24*b* becomes:

$$B = B_\infty \{1 - 3\, hv - \tfrac{3}{10}(hv)^2 - \tfrac{1}{18}(hv)^3 \ ...\} \ . \quad . \quad . \quad (34)$$

where B_∞ represents the value of B for $h = 0$, viz. for $T = \infty$.

From (34) we find for the BOYLE-point for which $B = 0$:

$$(hv)_B = 0{,}3223.$$

Writing now $t_{(B)}$ for $\frac{T}{T_B}$, where T_B is the BOYLE-temperature, so that $t_{(B)}$ represents the reduced temperature with respect to the BOYLE-point as reduction-temperature, we find for (34)

$$B = B_\infty \{1 - 0{,}9669\, t_{(B)}^{-1} - 0{,}0312\, t_{(B)}^{-2} - 0{,}0019\, t_{(B)}^{-3} ...\} \ . \quad (35)$$

[1]) Comp. H. KAMERLINGH ONNES and W. H. KEESOM. Die Zustandsgleichung. Leiden Suppl. N°. 23 § 38*b*.

while (30) becomes:

$$C = C_\infty \{1 - 0{,}457\, t_{(B)}^{-1} + 0{,}163\, t_{(B)}^{-2} \ldots\} \quad . \quad (36)$$

b. For $\varphi(r) = cr^{-5}$ we obtain:

$$B = B_\infty \{1 - \tfrac{3}{2} hv - \tfrac{3}{14} (hv)^2 - \tfrac{1}{24} (hv)^3 - \tfrac{1}{156} (hv)^4 \ldots\}, \quad . \quad . \quad (37)$$

$$(hv)_B = 0.6070$$

$$B = B_\infty \{1 - 0{,}9105\, t_{(B)}^{-1} - 0.0789\, t_{(B)}^{-2} - 0.0093\, t_{(B)}^{-3} - 0.0010\, t_{(B)}^{-4} \ldots\}. \quad (38)$$

$$C = C_\infty \{1 - 0{,}728\, t_{(B)}^{-1} + 0{,}298\, t_{(B)}^{-2} \ldots\}. \quad . \quad . \quad (39)$$

c. In fig. 1 B and C have been represented as functions of $t_{(B)}$ according to the above formulae, while B_∞, resp. C_∞ have been taken $= 1$.

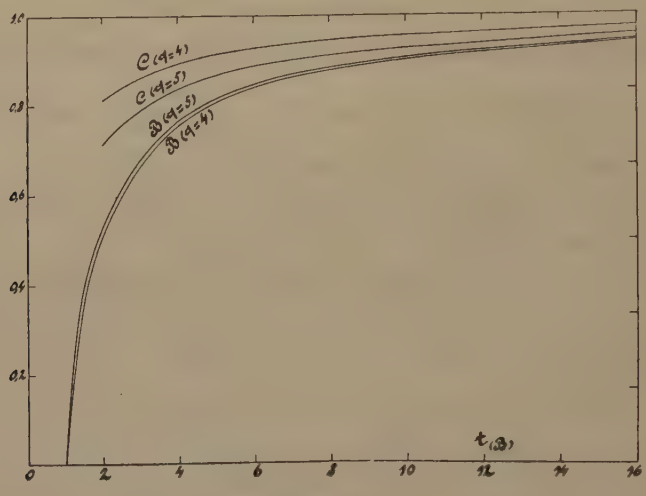

Fig. 1.

We see that within the represented range of temperature C is positive. C decreases with rising temperature, in a considerably less degree however than B.

How the behaviour will be towards lower temperatures it is impossible to say before more terms of the developments into series will have been calculated. When following terms do not change this result, the terms here calculated would cause a minimum of C, [1]) after which C would rise again, so that it is not improbable that C remains positive up to a domain of considerably lower temperatures than those to which fig. 1 refers.

While for the two laws of force considered the curves of B are nearly the same above the BOYLE-point, the coefficient C decreases with the temperature for $q = 5$ much more rapidly than for $q = 4$, at least in the beginning.

[1]) Comp. the representation of the values of C for helium in H. KAMERLINGH ONNES and W. H. KEESOM, Die Zustandsgleichung. Leiden. Suppl. N°. 23, § 38, Fig. 15.

Mathematics. — *"On the number of degrees of freedom of the geodetically moving system and the enclosing euclidian space with the least possible number of dimensions"*. By Prof. J. A. SCHOUTEN. (Communicated by Prof. J. CARDINAAL).

(Communicated in the meeting of May 25, 1918).

Suppose k to be a non-special curve in a finite part X_n of a general space of n dimensions, containing no singular points and where only one geodetic line exists between two arbitrary points. Assuming in a point O on k a system of n mutually independent directions, we can move this system *geodetically* along k.

This geodetic moving can be geometrically defined in the following way. X_n can always be placed in a euclidian space of $\frac{n(n+1)}{2}$ dimensions, without changing its linear element. There exists in this space a space Y_n developable on a euclidian space of n dimensions, tangent to X_n in k. The geodetically moving directions will now coincide at any moment with the directions moving parallel to themselves in the euclidian space Y_n. It appears analytically that the known *covariant* differential of a directed quantity e.q. a vector is a *common* differential judged from a geodetically moving system of directions. Hence if $\mathbf{v}$ is a vector stationary with respect to this system, $\mathbf{v}$ satisfies the differential-equation:

$$d\mathbf{v} = 0,$$

or in co-ordinates:

$$dv^\lambda + \left\{ \begin{matrix} \mu\nu \\ \lambda \end{matrix} \right\} v^\mu \, dx^\nu = 0,$$

and this equation then gives the analytical definition of the notion geodetic moving. [1]) A geodetic line is characterised by the property that its linear element forms at every point the same angles with a system moving geodetically along the line.

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[1]) The covariant notations in this paper are the customary ones, but the contravariant characteristic numbers of the linear element $d\mathbf{x}$ are written contravariant agreeing to G. HESSENBERG, but contrary to G. RICCI and T. LEVI CIVITA. For the invariant notations, the here used direct analysis, cf. "Ueber die direkte Analysis der neueren Relativitätstheorie", a paper presented to the "Koninkl. Akademie v. W." together with this note. (Verh. Vol. 12 N°. 6).

Starting from a point O a system of directions is now geodetically moved along a closed curve. On returning to O the system will generally appear to have rotated. Dependent on the choice of the curve it is generally possible to obtain in this manner $\infty^{\frac{n(n-1)}{2}}$ positions of the system. If this number is for one point and hence for every point of the area ∞^N, we call N the *number of degrees of freedom* of the geodetically moving system. Now the following theorem exists:

I. *The number of dimensions of the euclidian space, in which a given space X_n may be placed, without changing its linear element, is at most equal to the number of degrees of freedom belonging to the geodetically moving system increased with n.*

We will prove this theorem. If the number of degrees of freedom is smaller than $\frac{n(n-1)}{2}$, there will remain invariant r mutually perfectly perpendicular directions of $p_1, p_2, p_3, \ldots$ dimensions $p_1+\ldots+p_r=n$ (by direction of two dimensions or 2-direction we mean a plane direction, etc.). The number of possibilities exactly corresponds to the number of manners, in which n can be written as the sum of whole positive numbers. We imagine the r invariant directions marked once for all in O. The system then may be brought in every point of X_n, always by geodetically moving. The invariant p_j-direction, $j = 1 \ldots, r$, will then define at every point a p_j-direction, and it is the question whether these directions will compose a system of ∞^{n-p_j} curved spaces P_j of p_j dimensions. This is a PFAFFIAN problem in a general space.

We select a definite invariant direction, say the p_j-direction, and for convenience, sake we shall write p for p_j. If we now define the p-direction belonging to this direction at every point by the simple p-vectors, ${}_p\mathbf{v} = \mathbf{v}_1 \frown \ldots \frown \mathbf{v}_p$, which all pass into one another by geodetically moving and likewise the perfectly perpendicular $(n-p)$-direction to this, by ${}_q\mathbf{w} = \mathbf{w}_1 \frown \ldots \frown \mathbf{w}_q$, $q = n-p$, then:

$$d\,{}_p\mathbf{v} = 0 \quad , \quad d\,{}_q\mathbf{w} = 0,$$

hence:

$$\nabla\,{}_p\mathbf{v} = 0 \quad , \quad \nabla\,{}_q\mathbf{w} = 0.$$

It is worth mentioning, that the vectors $\mathbf{v}_k$, $k = 1, \ldots, p$, do *not* pass into each other by geodetically moving and hence $d\mathbf{v}_k \neq 0$. The same holds good for $\mathbf{w}_l$, $l = 1, \ldots, q$. If now the linear element

be $d\mathbf{x}$, the usual formulation of a problem as under consideration, is as follows.[1])

Given the pn functions of $x^1, \ldots, x^n$:

$$v_k{}^1, \ldots, v_k{}^n \quad ; \quad k = 1, \ldots, p$$

(the contravariant characteristic numbers of the vectors $\mathbf{v}$), and the qn functions:

$$w_{l_1}, \ldots, w_{ln} \quad ; \quad l = 1, \ldots, q$$

(the covariant characteristic numbers of the vectors $\mathbf{w}$), satisfying the relation:

$$\sum_{\lambda}^{1,\ldots,n} v_k{}^\lambda \, w_{l\lambda} = 0$$

equivalent to

$$\mathbf{v}_k \cdot \mathbf{w}_l = 0,$$

we ask, when the system of the total differential-equations

$$\begin{Vmatrix} dx^1 & \ldots & dx^n \\ v_1{}^1 & \ldots & v_1{}^n \\ \vdots & & \\ v_p{}^1 & \ldots & v_p{}^n \end{Vmatrix} = 0$$

equivalent to

$$d\mathbf{x} \; \mathbf{v}_1 \ldots \mathbf{v}_p = 0$$

is perfectly integrable.

If $\mathbf{r}$ and $\mathbf{s}$ are two vectors, lying in the p-vector ${}_p\mathbf{v}$, and consequently satisfying the relations:

$$\mathbf{w}_l \cdot \mathbf{r} = 0 \quad , \quad \mathbf{w}_l \cdot \mathbf{s} = 0 \quad , \quad l = 1, \ldots, q$$

which is equivalent to:

$$\sum_{\lambda}^{1,\ldots,n} w_{l\lambda} r^\lambda = 0 \quad , \quad \sum_{\lambda}^{1,\ldots,n} w_{l\lambda} s^\lambda = 0 \quad , \quad l = 1, \ldots, q,$$

but being otherwise arbitrary, the conditions of integrability are, as known:

$$\sum_{\mu,\nu}^{1,\ldots,n} \left(\frac{\partial w_{l\mu}}{\partial x^\nu} - \frac{\partial w_{l\nu}}{\partial x^\mu} \right) r^\mu s^\nu = 0 ; \quad l = 1, \ldots, q.$$

These equations are generally covariant[2]) and are equivalent to:

$$\mathbf{r} \frown \mathbf{s} \, ^2 \nabla \frown \mathbf{w}_l = 0 ; \quad l = 1, \ldots, q$$

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[1]) Cf. e.g. E. von Weber, Vorlesungen über das Pfaff'sche Problem, pages 93 and f.f.

[2]) Owing to the circumstance that the expression $\frac{\partial w_{l\mu}}{\partial x^\nu} - \frac{\partial w_{l\nu}}{\partial x^\mu}$ is generally covariant.

Now it follows from this mode of notation that they may be replaced by the invariant equation:

$$\mathbf{r} \frown \mathbf{s} \overset{2}{.} \nabla \frown (\mathbf{w}_1 \overset{\frown}{.} \ldots \mathbf{w}_q) = \mathbf{r} \frown \mathbf{s} \overset{2}{.} \nabla \frown {}_q\mathbf{w} = 0,$$

or, as follows from the preceding, still more simplified without making use of two auxiliary vectors:

$${}_p\mathbf{v} \overset{2}{.} \nabla \frown {}_q\mathbf{w} = 0.\ ^1)$$

But this equation is identically true, $\nabla \frown {}_q\mathbf{w}$ being a multiple-sum of isomers of $\nabla_q\mathbf{w}$, and $\nabla_q\mathbf{w}$ being zero.

As the plane tangent-spaces of p_j -dimensions, in the various points of the spaces P_j have p_j-directions, which by means of geodetic motion pass into each other and in the invariant p_j-direction in O, but never in any other direction, two spaces P_j can therefore never intersect. A geodetic line in X_n, which has a linear element in common with a definite space P_j, is apparently altogether contained in that space and in that space it is geodetic too. Hence two different spaces P_j can never be tangent to one another. Therefore we call the spaces P_j parallel ones. As any geodetic line having two points in common with a P_j space, falls completely within that space, which will be proved later on, we call a P_j space geodetic. The r obtained systems of parallel geodetic spaces $P_1, \ldots, P_r$ are at every point of X_n perfectly perpendicular to each other.

We shall first contemplate the case $r = 2$, $p_1 = p$, $p_2 = q$. The parameter-spaces of $n-1$ dimensions of the primitive variables $x^1, \ldots, x_p$ are placed thus that each of them contains ∞^{n-p-1} spaces P, those of $x^{p+1}, \ldots, x^n$ likewise with regard to the spaces Q. At every point we place the mutually perfectly perpendicular p- resp. q-vectors ${}_p\mathbf{v}$ and ${}_q\mathbf{w}$. The measure-vectors $\mathbf{e}'_\varkappa$, $\varkappa = 1, \ldots, p$ are then situated in ${}_p\mathbf{v}$ and for the measure-vectors $\mathbf{e}'_\mu$, $\mu = p+1, \ldots, n$ the same holds good with regard to ${}_q\mathbf{w}$. Because $\mathbf{e}'_\varkappa \perp \mathbf{e}'_\mu$ we have

$$g_{\varkappa\mu} = \mathbf{e}'_\varkappa \,.\, \mathbf{e}'_\mu = 0 \ ; \qquad \begin{matrix} \varkappa = 1, \ \ldots \ , p \\ \mu = p+1, \ \ldots \ , n \end{matrix}$$

hence the quadratic form ds^2 may be written thus:

$$ds^2 = \sum_{\varkappa,\lambda}^{1,\ldots,p} g_{\varkappa\lambda}\, dx^\varkappa\, dx^\lambda + \sum_{\mu,\nu}^{p+1,\ldots,n} g_{\mu\nu}\, dx^\mu\, dx^\nu$$

Now may be demonstrated, that $g_{\varkappa\lambda}$ is independent of $x^{p+1}, \ldots, x^n$, and likewise $g_{\mu\nu}$ is independent of $x^1, \ldots, x^p$. It is always possible to choose a scalar k as function of $x^1, \ldots, x^n$, so that:

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[1]) This equation can also be obtained very easily by means of the direct analysis used here. Another form of the same equation is:

$${}_q\mathbf{w} \overset{1}{.} (\nabla \overset{1}{.} {}_p\mathbf{v}) = 0.$$

$$_p\mathbf{v} = \mathbf{k}\,\mathbf{e}_1' \ldots \mathbf{e}_p$$

Consequently:

$$\nabla k\,\mathbf{e}_1' \ldots \mathbf{e}_\rho' = (\nabla k)(\mathbf{e}_1' \ldots \mathbf{e}_\rho') + k \sum_{\varkappa}^{1,\ldots,p} \nabla(\mathbf{a}\cdot\mathbf{e}_\varkappa')(\mathbf{e}_1' \ldots \mathbf{e}_{\varkappa-1}'\ \mathbf{a}\,\mathbf{e}_{\varkappa+1}' \ldots \mathbf{e}_\rho') = 0.$$

By complete transvection with:

$$\mathbf{e}_\rho' \ldots \mathbf{e}_{\varkappa+1}\ \mathbf{e}_\mu'\ \mathbf{e}_{\varkappa-1} \ldots \mathbf{e}_1\ \mathbf{e}_\nu'$$

all the terms except the $(\varkappa+1)$-th give zero, hence:

$$\left(\frac{\partial}{\partial x^\nu} a_\varkappa\right) a_\mu = 0 \qquad \begin{matrix}\varkappa = 1, \ldots, p \\ \mu, \nu = p+1, \ldots, n.\end{matrix}$$

Now:

$$g_{\mu\nu} = a_\mu\, a_\nu\ ,$$

and

$$\left(\frac{\partial}{\partial x_\nu} a_\varkappa\right) a_\mu = a_{\varkappa\nu}\, a_\mu = \begin{bmatrix}\varkappa\nu \\ \mu\end{bmatrix} = a_{\nu\varkappa}\, a_\mu,$$

thus

$$\frac{\partial g_{\mu\nu}}{\partial a^\varkappa} = a_{\mu\varkappa}\, a_\nu + a_\mu\, a_{\nu\varkappa} = 0.$$

Hence the linear element in the Q spaces is independent of $x^1, \ldots, x^p$; the corresponding property of the P spaces relative to $x^{p+1}, \ldots, x^n$ is similarly demonstrated.

This property can also be expressed in the following manner.

II. *If in a general space X_n is placed a system of ∞^{n-p} parallel geodetical spaces of p dimensions P, having perfectly perpendicular to it a system of ∞^p similar spaces Q of $n-p$ dimensions, a figure in a definite P-space will be congruently projected by the Q-spaces on all the other P-spaces.*

For $p=1$ this is the well-known property that the distance of two definite Q-spaces measured along the P-lines is constant. So we can here introduce in this case for primitive variable x^1 the curve length measured along these lines from a definite space, the spaces remaining parameter-spaces. Hence the linear element may then be expressed in the following way:

$$ds^2 = dx^{1^2} + \sum_{\mu,\nu}^{2,\ldots,n} g_{\mu\nu}\, dx^\mu\, dx^\nu$$

in which the $g_{\mu\nu}$ do not depend on x^1 [1]).

As, however, a quadratic differential form in $n-1$ variables can

[1]) This formula has already been derived by T. Levi Civita. Nozione di parallelismo in una varietà qualunque e consequente specificazione geometrica della curvatura Riemanniana. Rend. di Pal. 42 (17).

always be written as a sum of $\frac{n(n-1)}{2}$ quadrates of complete differentials, ds^2 can be reduced to a sum of $\frac{n(n-1)}{2}+1$ similar quadrates. Hence the space X_n can be placed in this case in a euclidian space of

$$\frac{n(n-1)}{2}+1=\frac{(n-1)(n-2)}{2}+n$$

dimensions. As the number of degrees of freedom of the geodetic moving system amounts exactly to

$$\frac{(n-1)(n-2)}{2},$$

the required proof has been furnished.

If we now return to the case $r=2$, $p_1=p$, $q_2=q$, the number of degrees of freedom is $\frac{p(p-1)}{2}+\frac{q(q-1)}{2}$. The quadratic form breaks up into two forms, which may be written as a sum of $\frac{q(q+1)}{2}$ resp. $\frac{p(p+1)}{2}$ quadrates. Therefore the space X_n can be placed within a euclidian space of

$$\frac{p(p+1)}{2}+\frac{q(q+1)}{2}=\frac{p(p-1)}{2}+\frac{q(q-1)}{2}+n$$

dimensions, and here again the required proof has been furnished.

The case $q>2$ may be reduced to the preceding one. For this purpose the differential form is divided into two. The spaces of one of the systems, say P_1, then again contain themselves at least two perfectly perpendicular systems of parallel geodetic spaces. Then the second part of the differential form is once more divided etc.

If owing to the existence of the P_j-system the division of the differential form is:

$$ds^2=\mathbf{a}^2\,d\mathbf{x}^2=\mathbf{a}_p^2\,d\mathbf{x}^2+\mathbf{a}_q^2\,d\mathbf{x}^2,$$

in which $\mathbf{a}_p$ and $\mathbf{a}_q$ are the ideal radices of the two parts of the fundamentaltensor $\mathbf{a}^2$, the differential equation of a geodetic line will be:

$$d\frac{d\mathbf{x}}{ds}=d\left(\mathbf{a}\cdot\frac{d\mathbf{x}}{ds}\right)\mathbf{a}=d\left(\mathbf{a}_p\cdot\frac{d\mathbf{x}}{ds}\right)\mathbf{a}_p+d\left(\mathbf{a}_q\cdot\frac{d\mathbf{x}}{ds}\right)\mathbf{a}_q=0.$$

$\mathbf{a}_p$ compossing itself only of the measure vectors $\mathbf{e}_1', \ldots, \mathbf{e}_p'$ and $\mathbf{a}_q$ only of $\mathbf{e}_{p+1}', \ldots, \mathbf{e}_n'$, we have:

$$d\left(\mathfrak{a}_p \cdot \frac{d\mathfrak{x}}{ds}\right)\mathfrak{a}_p = 0 \quad , \quad d\left(\mathfrak{a}_q \cdot \frac{d\mathfrak{x}}{ds}\right)\mathfrak{a}_q = 0,$$

from which the property is inferred:

III. *In the proposition made in formulating II the projection of a geodetic line by means of Q spaces on a P space, or vice versa, is, as far as existent, a geodetic one itself.*

If two points A and B are situated in a P space P_1, the projections of these points on all spaces coincide. Hence the projections on a Q space of the line AB geodetic in X_n passes twice through the same point, being at the same time geodetic in Q, which is only possible when that projection has degenerated into a point. But then the geodetic line AB must be situated altogether in a P space, e.g. in the present case in P_1.

Hence any geodetic line, having two points in common with a P-space, is entirely contained in that space.

Mechanics. — *„Bemerkungen über die Beziehungen des* DE SITTER*'schen Koordinatensystems B zu der allgemeinen Welt konstanter positiver Krümmung."* By Prof. FELIX KLEIN.

(Communicated in the meeting of September 29, 1918).

1. Ich gehe zunächst von dem sphärischem Falle aus und setze:

$$\xi^2 + \eta^2 + \zeta^2 - v^2 + \omega^2 = R^2,$$
$$ds^2 = \pm (d\xi^2 + d\eta^2 + d\zeta^2 - dv^2 + d\omega^2),$$

wo das $+$ oder $-$ Zeichen gelten soll, je nachdem wir einen „Räumlichen" oder „Zeitlichen" Vector messen. Der Uebergang zum elliptischem Falle erfolgt am einfachsten, indem wir setzen:

$$x = \frac{R\xi}{\omega}, \quad y = \frac{R\eta}{\omega}, \quad z = \frac{R\zeta}{\omega}, \quad u = \frac{Rv}{\omega}.$$

2. Dann will ich von vorn herein betonen, dan die so definierte Welt bei 10^{10} homogenen linearen Substitutionen der ξ, η, ζ, v, ω — also bei ∞^{10} Kollineationen der x, y, z, u (welche $x^2 + y^2 + z^2 - u^2 + R^2 = 0$ in sich überführen) — in sich übergeht.

3. Den Uebergang zum DE SITTER'schen ds^2 mache ich nun so, dass ich setze:

$$\xi = R \sin \vartheta \cos \varphi,$$
$$\eta = R \sin \vartheta \sin \varphi \cos \psi,$$
$$\zeta = R \sin \vartheta \sin \varphi \sin \psi,$$
$$v = R \cos \vartheta \sin h \frac{ct}{R},$$
$$\omega = R \cos \vartheta \cos h \frac{ct}{R},$$

(unter $\sin h$, $\cos h$ gewöhnliche hyperbolische Funktionen verstanden).

4. Weiter wird also

$$\tan h \frac{ct}{R} = \frac{v}{\omega} = \frac{u}{R},$$

woraus

$$\frac{ct}{R} = \log\left(\frac{\omega + v}{\omega - v}\right) = \log\left(\frac{R + u}{R - u}\right).$$

Dabei sind $\omega + v = 0$, $\omega - v = 0$ irgend zwei Tangential-ebenen, welche man durch den Koordinatenanfangspunkt an dem Gebilde $\xi^2 + \eta^2 + \zeta^2 - v^2 + \omega^2 = 0$ legen kann; t ist der in geeigneter Maassseinheit gemessene „projektive Winkel" im Büschel

$$\lambda(\omega + v) + \mu(\omega - v) = 0.$$

5. In Uebereinstimmung damit wird $t = -\infty$, bezw. $t = +\infty$, wenn $\omega + v = 0$ oder $\omega - v = 0$; verschwinden beide Ausdrücke, so wird t unbestimmt[1]). *t ist nur in dem Ausschnitte der vierdimensionalen Welt reell, für den $\frac{\omega + v}{\omega - v}$ positiv ist; jenseits wird es imaginär.*

6. ds^2 behält seine Form bei ∞^4 Transformationen, nämlich wenn man

a) ξ, η, ζ beliebig orthogonal substituiert,

b) t durch $t + const.$ ersetzt.

Mit Rücksicht auf 2 gibt es also in der Welt (1) ∞^6 „DE SITTER'schen Uhren".

7. Da ist es nun sehr amüsant sich auszumalen, wie sich zwei Beobachter unterhalten würden, die mit zwei verschiedenen DE SITTER-schen Uhren ausgestattet sind. Ereignisse, die für den einen in der Ewigkeit liegen, sind für den anderen zugänglich, und umgekehrt, ja der eine erlebt Ereignisse, welche der andere für imaginär hält.

8. Dies alles sind zunächst selbstverständlich nur mathematische Spekulationen, ob der Physiker sie aufnehmen will steht auf einem anderem Blatte.

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[1]) Dies ist aber nur eine Singularität der Koordinate t, nicht der zu Grunde liegenden Mannigfaltigheit (1).

Physics. — *"On friction in connexion with Brownian movement"*. By Dr. O. POSTMA. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the meeting of September 29, 1918).

§ 1. The different deductions of the mean displacement of a suspended particle caused by the Brownian movement in a certain time, can be divided into two groups. In the deductions of the *first* kind the forces acting on the particle are divided into accelerating and retarding forces, in those of the *second* kind this difference is not made.

To the deductions of the first group belongs that of EINSTEIN, the second one of VON SMOLUCHOWSKI, that of LANGEVIN and that of DE HAAS—LORENTZ (by the method of EINSTEIN and HOPF). To those of the second group belongs the first deduction of VON SMOLUCHOWSKI, that of VAN DER WAALS and SNETHLAGE and that of SNETHLAGE (given in her dissertation) [1] [2]).

In the theories first mentioned a retarding force (friction) has been assumed proportional to the radius of the particle, which is considered as a sphere and has the velocity v. The applied formula $W = 6\pi\zeta av$ is deduced in hydrodynamics on the supposition that at the surface of the particle the fluid has the same velocity as this. All theories of the first kind come to the result that the mean square of the displacement ($\overline{\Delta^2}$) is inversely proportional to t and proportional to a.

The theories of the second kind give the result: $\overline{\Delta^2}$ proportional to t and inversely proportional to a^2 and to the density ϱ [3]).

It is however not to the resolution of the forces into accelerating

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[1]) See: G. L. DE HAAS – LORENTZ "Die Brownsche Bewegung", die Wissenschaft 1913, J. D. VAN DER WAALS JR. and Miss A. SNETHLAGE: "The theory of the Brownian movement". These Proceedings Vol. XVIII, p. 1322 and A. SNETHLAGE: 'Molecular-kinetic phenomena in gases, especially the Brownian movement". Amsterdam 1917.

[2]) Recently VAN DER WAALS JR. gave still a theory in which, though in a somewhat different form, the two kinds of forces are found back. (These Proc. XX p. 1254.

[3]) The first result gives the best agreement with the investigations of SNETHLAGE.

and retarding ones that these different results must be ascribed. This is evident from considerations of VON SMOLUCHOWSKI, who, though assuming such a resistance proportional to v, obtained the second results by simply taking this resistance proportial to a^2 and ϱ.

The reason of the difference lies in the assumptions on the sliding at the surface. Without sliding the resistance is found proportional to a, and not to a^2 and ϱ. As long as the velocities of the molecules may be considered as independent of those of the particle, the assumption W proportional to $\varrho a^2 v$ is the more natural one.

The number of collisions is then proportional to the surface of the particle and the density, while the total force is also proportional to v by the added opposite influences of the forces on the fore- and back-side, each of which is proportional to the square of the relative velocity. This resistance has been calculated by CUNNINGHAM [1]), who found: $\frac{8}{3} a^2 v n \sqrt{\frac{\pi M m}{(M+m)h}}$.

When however the velocity of the molecules depends on that of the particle the relative velocity v can again depend on a, so that the dependency on a becomes different from that in the preceding formula.

It is therefore evidently not true that the assumption of a resisting force, represented by $w = pv$ should be connected with the supposition that there is no sliding.

It is the last assumption that is of importance here.

VAN DER WAALS and SNETHLAGE have objections to this assumption. When the molecules of the fluid participate in the motion of the particle, the velocities of the molecule are no longer independent of the velocity of the particle, which should however be the case in the here existing state of statistic equilibrium. This participation of the motion would e.g. be possible in the case of the fall of a particle under the influence of gravitation, but not here in that of the BROWNIAN movement.

Here it may be remarked however, that just by the resolution of the forces into accelerating and retarding ones an analogous case has been created. The accelerating forces take the place of gravitation. The action of the molecules on the particle is resolved into an accidental, irregular one, which may be regarded as the cause of the motion and a regular, resisting one which represents the friction.

When we consider *a long time* there is statistical equilibrium. There is however a continual transformation of energy of the

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[1]) E. CUNNINGHAM: "On the Velocity of Steady Fall of Spherical Particles through Fluid Medium". Proc. Roy. Soc. Ser. A. Vol. 83 p. 357. 1910.

molecules by the irregular, accelerating forces into molar energy of the particle, and of this again by the resisting forces into molecular energy.

Also the molecular velocities are *as a whole* independent of those of the particle, but the molecules can be divided into a large majority the mean velocity of which differs too little from that of the particle, and into smaller complexes whose velocities differ too much from that. The first group joins the motion of the particle, the second causes the accelerating forces. These forces, however, are not only caused by accidental deviations of the velocity, but also by accidental deviations of density. The relative dimensions of the molecules, the particles and the free path agree with the above considerations. While the order of magnitude of the molecules is 10^{-7}, we generally find for that of the particles about 10^{-5}, the free path in the fluid being of the order 10^{-6}. We thus can represent the dimensions of the molecules and the particles on a scale by 1 mm and 1 dm, while then the free path would become 1 cm. As a rough approximation we may assume that in a fluid 10^4 molecules are acting at the same time on the particle (of the same order of magnitude as the number that can cover the whole surface) and in a gas 10^2. The collisions which keep the particle in motion are therefore due to accidentally arising complexes of these molecules.

§ 2. Now we find for the mean distance travelled by a particle EINSTEIN's formula $\overline{\Delta^2} = \frac{RT}{N} \frac{1}{3\pi\zeta a} . t$, when we take for the frictional resistance $W = 6\pi\zeta a v$. This value has been deduced by STOKES on the assumption that the fluid is incompressible and that there is not any sliding at the surface, while the velocity may not be too great. It is however questionable, whether it is a allowed to assume this absence of sliding.

CUNNINGHAM has tried the supposition that the velocity of the fluid at the surface of the particle is kv, so that the relative velocity of the particle with respect to the adjoining fluid would be $v - kv$. STOKES' formula would then give for the resistance $W = 6\pi\zeta a\, kv$. CUNNINGHAM now calculated this k in the following way.

As has been mentioned above he found for the resistance by purely kinetic considerations the value $\frac{8}{3} a^2 v n \sqrt{\frac{\pi M m}{(M+m)h}}$; where v represents the relative velocity. This would now become

$\frac{8}{3}a^2(v-kv)\sqrt{\frac{\pi m}{h}}$ (when at the same time m is neglected with respect to M). He now equalizes this value to the value $6\pi\zeta akv$ derived from phenomenological considerations, and thus finds an equation from which k can be calculated.

In my opinion, however, there are several objections to this way of calculation. The assumptions on which the deductions of the two expressions are based are not the same; the results refer therefore to different cases and need not be equal. In his kinetic considerations CUNNINGHAM supposed namely, that the colliding molecules possess MAXWELL's partition of velocities, viz. in the gas or the fluid there is no internal friction (this depends on the deviations from MAXWELL's partition), while moreover colliding molecules exert only normal forces on the particles viz. there is no external friction.

In the case of STOKES' formula on the contrary there is an internal friction and the fluid exerts also lateral forces on the body. In my opinion it is therefore impossible by equalization of the two results to obtain a relation which has any significance. Moreover there is still the inner contradiction in the kinetic deduction between a uniform transformation of velocity kv and the assumption of a purely elastic collision. For this reason no great importance may be ascribed to the final result: $W = 6\pi\zeta av\left(1 + 1{,}63\frac{l}{a}\right)^{-1}$.

Instead of assuming a uniform motion of the fluid with the sphere to an amount kv it might be preferable to suppose the normal velocity of the fluid to be equal to that of the particle, the tangential one being different and a friction existing proportional to the relative tangential velocity.

The resistance of a sphere under this assumption has been calculated by LAMB and by BASSET[1]). The first of these uses in his deduction the dissipation function (with the aid of the property: in a fluid the energy transformed into heat = the work of the forces necessary to entertain the motion); the other one follows a direct method by calculating the pressure at the surface. The two results however do not agree exactly. LAMB finds: $W = 6\pi\zeta av\frac{1 + 4\frac{\zeta}{\beta a} + 6\left(\frac{\zeta}{\beta a}\right)^2}{\left(1 + 3\frac{\zeta}{\beta a}\right)^2}$,

[1]) See: HORACE LAMB "A Treatise on the Math. Theory of the Motion of Fluids". Cambr. 1879 p. 230 (not reprinted in the later issues) and A. B. BASSET, "A Treatise on Hydrodynamics" Cambr. 1888 II p. 270.

and BASSET: $W = 6\pi\zeta a v \dfrac{1+\dfrac{2\zeta}{\beta a}}{1+\dfrac{3\zeta}{\beta a}}$, where β is the coefficient of external friction. The second expression is also equal to: $6\pi\zeta a v \dfrac{1+5\dfrac{\zeta}{\beta a}+6\left(\dfrac{\zeta}{\beta a}\right)^2}{\left(1+3\dfrac{\zeta}{\beta a}\right)^2}$, so that the difference between the results is: $6\pi z\zeta v \dfrac{\dfrac{\zeta}{\beta a}}{\left(1+3\dfrac{\zeta}{\beta a}\right)^2}$.

A mistake suggests itself in the indirect method and seeking for it we are led by the remark that the two values become equal when $\beta = 0$ and when $\beta = \infty$ viz. when there is no friction and when there is no sliding.

In both cases the friction does no work: in the first case because the force becomes zero, in the second because the way becomes zero.

Probably we shall therefore have to add the work of the friction at the surface to the heat, calculated from the dissipation-function with the formula $\iiint F\,dx\,dy\,dz$, where $F = -\frac{2}{3}\zeta(a+b+c)^2 + 2\zeta(a^2+b^2+c^2+2f^2+2g^2+2h^2)$ [1]. This proves indeed to give the right result.

At a point determined by the angle θ the force of friction is: $U = \beta\left\{\dfrac{A}{a^2} - \dfrac{B}{a} + v\right\}\sin\vartheta$ [2]), where $A = \dfrac{-\frac{1}{4}va^2}{1+3\dfrac{\zeta}{\beta a}}$ and $B = \dfrac{\frac{3}{4}va\left(1+\dfrac{2\zeta}{\beta a}\right)}{1+3\dfrac{\zeta}{\beta a}}$ so that $U = \beta\dfrac{\dfrac{3\zeta v}{2\beta a}}{1+\dfrac{3\zeta}{\beta a}}\sin\vartheta$. This friction is exerted on the surface $2\pi a^2 \sin\vartheta\, d\vartheta$, while the relative velocity, opposite to the force, is $\dfrac{\dfrac{3\zeta v}{2\beta a}}{1+\dfrac{3\zeta}{\beta a}}\sin\vartheta$.

[1]) Notation of LAMB.

[2]) See LAMB p. 230 and BASSET p. 270.

We thus find for the numerical value of the work of the friction per second:

$$\int_0^\pi 2\pi a^2 \sin\vartheta\, d\vartheta\, \beta \left[\frac{\frac{3\zeta v}{2\beta a}}{1+\frac{3\zeta}{\beta a}} \sin\vartheta\right]^2 = \int_0^\pi 2\pi a^2 \sin^3\vartheta \frac{\frac{9\zeta^2 v^2}{4\beta^2 a^2}}{\left(1+\frac{3\zeta}{\beta a}\right)^2} d\vartheta =$$

$$= 6\pi\zeta v^2 a \frac{\frac{\zeta}{\beta a}}{\left(1+\frac{3\zeta}{\beta a}\right)^2}.$$

When to this term the heat calculated from the dissipation-function is added, the force, necessary to entertain the motion, and therefore also the friction in question, is just increased by the missing term.

With the aid of this last formula for the resistance we find for the mean deviation of the particles:

$$\overline{\Delta^2} = \frac{RT}{N} \cdot \frac{1}{3\pi\zeta a} \cdot \frac{1+\frac{2\zeta}{\beta a}}{1+\frac{3\zeta}{\beta a}} \cdot t.$$

§ 3. That, when the above mentioned work of the resistance is taken into consideration, the dissipation function can give the resistance to which the particle is submitted, may be made evident in the following way.

We think the incompressible fluid enclosed by a surface S, part of which is formed by the surface of a body of arbitrary form, which moves through the fluid. Now we consider the kinetic energy $T = \frac{1}{2}\iiint \varrho q^2\, dx\, dy\, dz$ (the molar energy only, not the heat) and deduce from the equation of motion an energy equation which indicates how the kinetic energy changes with the time. We then find [1]):

$$\frac{dT}{dt} = \iiint \varrho(Xu+Yv+Zw)\, du\, dy\, dz - \tfrac{1}{2}\iint \varrho q^2 (lu+mu+nw)\, dS +$$

$$+ \iint (X_1 u + Y_1 v + Z_1 w)\, dS - \iint F\, dx\, dy\, dz. \quad . \quad . \quad . \quad (1)$$

Here X, Y, Z represent the components of the external force acting on the fluid, pro unit of mass, X_1, Y_1, Z_1 the pressure components

[1]) See Basset II l.c. p. 252 and Helmholtz "Wissenschaftl. Abhandlungen" Lpzg. 1882 p. 225.

pro unit of surface acting at the surface of the fluid, F the dissipation function. The first term on the right hand side represents thus the increase of the kinetic energy by the action of external forces, the second term the increase by the flow inwards of the fluid, the third term the increase by the action of the pressure forces, the fourth the decrease by the transformation of the kinetic energy by the inner resistance. This last integral is always positive, the other ones can be positive or negative.

When we suppose the surface S to be quite formed by solid bodies, at the boundaries of which the friction is neglected except along the moving body, the second term falls away, while the third term has to be taken over this body only. When further the state is stationary, equation (1) becomes:

$$0 = \iint (X_1 u + Y_1 v + Z_1 w)\, dS - \iiint F\, dx\, dy\, dz \quad . \quad . \quad . \quad (2)$$

For the moving body we have in this case the equation:

$$\Sigma (X' u + Y' v + Z' w)\, dt + \iint (-X_1 u - Y_1 v - Z_1 w)\, dS\, dt = 0 \quad . \quad (3)$$

The summation has to be extended over the points, where the external forces (X' etc.) act, the integration over the surface. This notation expresses that the forces acting on the surface layer are equal and opposite to those acting there on the fluid.

In the case of sliding however the components of velocity are not equal as far as they refer to the body, on one hand, and to the fluid on the other hand. Calling the components in the fluid u_v etc. those in the body u_l etc. then $u_v = u_l + u_r$ etc. (4), when u_r is written for the component of the relative velocity.

Now equation (2) becomes:

$$\iint (X_1 u_v + Y_1 v_v + Z_1 w_r)\, dS - \iiint F\, dx\, dy\, dz = 0$$

and equation (3):

$$\Sigma (X' u + Y' v + Z' w) - \iint (X_1 u_l + Y_1 v_l + Z_1 w_l)\, dS = 0.$$

Summation gives us:

$$\Sigma (X' u + Y' v + Z' w) +$$

$$+ \iint [X_1 (u_v - u_l) + Y_1 (v_v - v_l) + Z_1 (w_v - w_l)]\, dS - \iiint F\, dx\, dy\, dz = 0$$

so, using (4):

$$\Sigma (X' u + Y' u + Z' w) = \iiint F\, dx\, dy\, dz - \iiint X_1 u_r + Y_1 v_r + Z_1 w_r)\, dS.$$

As the normal forces at the surface are perpendicular to the relative velocity they do not contribute to the last integral; this term represents therefore the work of the friction. The result is thus: the work done per unit of time by the external forces that act on the body is equal to the heat generated in the fluid increased by the absolute value of the work of the external friction [1]).

Taking now as the body a sphere with the velocity v, which is kept in a state of uniform motion by a force K in the centre, the work per unit of time is Kv. This K being equal to the resistance W that has to be overcome, it is also eqnal to

$Wv = \iiint F\,dx\,dy\,dz +$ the work done by the friction along the surface. From this formula we can calculate W.

[1]) "Absolute" value, as X_1 and u_r referring both to the fluid are oppositely directed, so that finally the form becomes positive. The result says therefore only: external work = total generated heat. It must be remarked, that RAYLEYGH, who first introduced the dissipation function, did not mean the above mentioned F, but the total generated heat. (Proc. Lond. Math. Soc. 1873 p. 363).